An Enquiry Concerning The Hyperreal Number System

By

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Abstract

The *hyperreal number system* has been developed since the 1960s, starting with the ground-breaking work of Abraham Robinson, for the purpose of doing *nonstandard analysis* [10]. The main motivation for its development derives from the goal to conceptualise historically controversial numbers, namely *infinitesimals* and *infinities*. In this thesis, my goal is first of all to provide a complete but concise construction of the hyperreal number system that functions as the basic conceptual frame of *nonstandard analysis*, and second of all to investigate some fundamental properties of the system that determines the way in which *nonstandard analysis* can be conducted. This thesis can be considered as an original synthesis primarily of [5], [4], and [10].

Introduction: Infinitesimals, Hyperreal Numbers, And The Nonstandard Analysis

From a historical point of view, the notion of *hyperreal numbers* is derived from the much older and controversial notion of infinitesimals. While historically discussions and applications of infinitesimals have appeared in various contexts where different understandings and characterisations of infinitesimals have been put forth, the original conception of infinitesimal can be traced back to Ancient Greece. It is believed that Archimedes came up with the formula for the area of a circle through the basic intuition that a circle can be viewed as a polygon with infinitely many sides that are infinitesimals in length [4].

Despite its somewhat convoluted formulation (which will be discussed shortly), the notion of infinitesimals has been invoked to develop many significant theories, including the invention and development of Calculus originally formalised by Gottfried Leibniz and Isaac Newton in the 17th century. Although the employment of such novel concept at the moment did help to bring in revolutionary progress in solving problems encountered in both differential and integral Calculus, these proponents of infinitesimals have failed to supply a rigorous description of the nature of this special category of numbers. Indeed, despite its wide potential applications, infinitesimals

have largely remained a controversial concept due to the lack of coherency in their formulation. In fact, criticisms against infinitesimal as a coherent concept have been raised by many. In particular, there are notable objections put forth by Georg Cantor and George Berkeley (Details can be found in [4], [8]). What is more, at the end of 19th century, because of its problematic connotation, the notion of infinitesimals was almost entirely exiled from the mainstream practice of mathematics in the western world. It was predominantly replaced by the new concept of *limits* that was developed by prominent mathematicians such as Cantor, Cauchy, and Weierstrass.

It was not until the 1960s when Abraham Robinson eventually managed to develop the first rigorous account of infinitesimals through the construction of the *hyperreal number system*, a number system that contains infinitesimals as well as infinities, that the notion of infinitesimals was accepted and revived by the mathematics community as a coherent and workable mathematical concept. Since then, the mathematical analysis that is based on the hyperreal number system has been recognised officially as *Nonstandard Analysis (or NSA)*.

To help us understand better the dramatic and troublesome history of infinitesimals, let us briefly investigate their traditional conception. According to the rich account provided by the *Stanford Encyclopedia of Philosophy*, Infinitesimals, commonly known as the infinitely small entities, can be understood as either magnitudes or numbers. On the one hand, insofar as a magnitude, an infinitesimal can be viewed as the "ultimate component" of a *continuum*, i.e. an infinitely divisible entity such as the space of real numbers, that constitutes the continuum. On the other hand, insofar as a number, an infinitesimal can be viewed as a number that is smaller than all numbers measurable but bigger than zero [1].[1] To clarify, no matter if they are viewed as magnitudes or numbers, infinitesimals are granted with a very special status that distinguishes them from all other magnitudes or numbers—namely, they

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[1]Notice that only positive numbers are concerned here.

are understood as the most basic building blocks of any magnitude or number.

One natural question that could be raised concerning such special status of infinitesimal is: if an infinitesimal itself is a magnitude, is the infinitesimal thus self-composed? Alternatively, if an infinitesimal itself is a number, is the infinitesimal thus beyond any measurement? Intuitively speaking, the answer to the question should be positive, for, otherwise, absurdity would be necessarily implied. To explain, if an infinitesimal is not self-composed, then it must be composed by some other magnitude, which would imply the existence of even more basic magnitudes. Alternatively, if an infinitesimal is indeed measurable, then it must be smaller than itself. Therefore, in either sense, the negative response to the question would lead to some absurd consequences. Hence, infinitesimals under this interpretation have to be self-composed (also known as indivisible) and unmeasurable. Observe that these features of infinitesimals (as the necessary consequences of the traditional understanding) have been deemed as rather unsatisfactory and have become the main cause of doubts and objections. To clarify, because infinitesimals are indivisible and unmeasurable, it seems impossible for one to observe or conceive them in the actual world. As a result, infinitesimals seem to only exist as some mysterious and unintelligible entities. Hence, such understanding of infinitesimals have been rendered as inadequate to sustain any solid ground for developing theories such as Calculus.

In this thesis, my main goal is to provide a succinct picture of the hyperreal number system, and to show how infinitesimals, insofar as a special category of hyperreal numbers, not only can be measured but also can be divided into more nuanced sub-categories. Specifically, I will employ three chapters to carry out this goal. In the first chapter, I will provide the basic tools and concepts from mathematical logic.[2] In the second chapter, I will provide a construction of the hyperreal number system that

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[2]Concerning the basic methodology of this thesis, notice that since the role of first order logic (a crucial concept from mathematical logic that will be clarified substantially in the next chapter) would function as the language of the discourse, this method thus is also considered as a point of view from the foundation of mathematics.

is originally due to Robinson [5], [4] and [10]. In the third chapter, I will investigate and synthesise the various properties of the new system from perspectives primarily in topology, set theory, and abstract algebra.

In the rest of this introduction, I will then make a few basic clarifications that perhaps would bring in some slightly more concrete understanding of hyperreal number system before delving into the comprehensive construction. Specifically, I will introduce one rigorous way to define infinitesimals and infinities that is due to Goldblatt [4].[3] Accordingly, given the set of all natural numbers $\mathbb{N}$, an infinitesimal can be defined as a number ϵ such that:

$$\forall n \in \mathbb{N}, \epsilon < \frac{1}{n}.$$

And an infinity can be defined as a number ω such that:

$$\forall n \in \mathbb{N}, n < \omega.$$

Observe that this definition sheds lights on the relationship between the real number system and the hyperreal number system. On the one hand, no real numbers as a result of this definition could qualify as either infinitesimal or infinity due to the *Eudoxus-Archimedes Principle* (which will be investigated in depth in chapter 2). On the other hand, any positive number that is smaller than all real numbers would therefore qualify as infinitesimals and any numbers that are bigger than all real numbers qualify as infinities. Consequently, *it is precisely in virtue of the failure of the real number system to incorporate infinitesimals and infinities that the hyperreal number system stands out as a more inclusive system, and hence it can be viewed as an extension of the real number system.*

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[3]This definition will be recognised consistently throughout the rest of the thesis.

Chapter 1

Tools From First Order Logic: Language And Structure

To some extent, the practice of mathematics has been seen as an activity to develop knowledge of mathematical entities through theories of properties and relations. In order to engage in mathematics, mathematicians have to establish their own language system that functions as the basic framework to conduct their discourse. Over the history of mathematics, its language has gone through considerable changes. Despite the high diversity of language, the objects of the discussion remain relatively stable. Some mathematical entities such as geometric shapes and numbers have been at the centre of mathematical endeavour since the birth of mathematics (in a way, the inquiry around these two concepts defines the very beginning of the subject). During the contemporary era, more specifically since the development of set theory and symbolic logic at the beginning of 20th century, mathematical language has steadily shifted to the system known as *first order logic*. In fact, this language system will be utilised as the framework for our construction of the hyperreal number system.

1.1 Language

The first order language as a syntax structure starts with some atomic formulae: namely, some n-tuple relations denoted as $R_n()$,[4] that is, some relation on a finite number of constants $\{a, b, c...\}$ and/or variables $\{x, y, z...\}$ that are used to represent collections of constants through quantifiers (the definition and use of quantifiers will be clarified). Notice that despite this perhaps seemingly simplistic set-up of the language system—where the only basic machinery is *relation*, one powerful feature of the concept of relation is that relations can be used to potentially predicate over any finite number of objects, namely, either constants or variables. As it will be shown, this feature is able to provide a solid basis to expand the expressibility of first order language considerably. To give a few examples: all sets are equivalent to some unitary relation—to say *S(x)*, meaning x is in S and thus a relation per se. Another categories of examples include basic binary relations, $<, =, >$, that exist in any ordered number system, such as the integer system, the rational number system, and so forth.

Before we move to finish building the structure of the language system in first order logic, let's consider one generic example of an atomic formula that predicates over three objects: R_3 (a, b, c). In other words, this formula amounts to saying: the constant objects a, b and c altogether form such a relation (or are in such a relation) called R_3. Note that at this point one cannot give an example of some concrete mathematical relation such as the binary ordering relations, since the language as the syntactic structure insofar as some pure formalism is void of concrete meaning. In other words, a relation of some objects in a language that has not been interpreted are simply some abstract symbols, which would need to be instantiated with particular meaning and thus truth value. As we will see in the next section, it is only under the interpretation of certain semantics, known as the *structure* of a language, that a

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[4]Notice that the use of subscript for relations is arbitrary. In particular, although it is often the case for convenience, the subscript "n" of the n-tuple relation does not necessarily entail the number of objects predicated by the relation.

formula in the form of first order language can acquire its meaning and acquire its definite truth value as a result.

Based on the concept of atomic formula as the basic building block of first order language, a well-formed formula (*wff*) is defined as a composition of atomic formulae connected by quantifiers ($\forall$,$\exists$), connectives ($\wedge$, $\vee$, $\equiv$, $\rightarrow$, and $\neg$), and/or brackets $((,),[,],\{,\})$. For example,

$$\forall(x,y)\ [R_2(a,x) \wedge R_3(y,b,c)].$$

In other words, this *wff* says that for all variable xs and ys, x is in relation R_2 with constant a, and y is in relation R_3 with constant b and c.

One crucial point that should be clarified here is: despite the syntactic nature of language that I have stressed above, the nature of the connectives and quantifiers that are required to build up *wff*s as part of the complete syntax of this language are semantic. In other words, in order to acquire a definite truth value of a formula in a language, we need to start with clarifying how connectives and quantifiers determine the truth value the formula. Specifically, all the connectives are essentially concrete relations themselves that are either unitary or binary with certain definite truth value. Here are the details:

1. "$\forall$": reads as "for all", and quantifies over all of its target variables. For example, the formula:

$$\forall(x)[R_1(x) \rightarrow R_1'(x)]$$

is true only if all possible xs satisfy the formula inside the brackets.

2. "$\exists$": reads as "there exists", and it quantifies over at least one of its target variables. For example, the formula:

$$\exists(x)[R_1(x) \rightarrow R_1'(x)]$$

is true only if at least one of possible xs satisfies the formula inside the brackets.

3. "$\wedge$": reads as "and". It is a binary relation such that, given two formulae (atomic or *wff*): p, q,

$$p \wedge q \text{ is true when both } p \text{ and } q \text{ are true.}$$

4. "$\vee$": reads as "or". It is a binary relation such that, given two formulae (atomic or *wff*): p, q,

$$p \vee q \text{ is true when either } p \text{ or } q \text{ is true or both } p \text{ and } q \text{ are true.}^{[5]}$$

5. "$\rightarrow$": reads as "implies". It is a binary relation such that, given two formulae (atomic or *wff*): p, q,

$$p \rightarrow q \text{ is always true except when } p \text{ is true and } q \text{ is false (when a true formula}$$
$$\text{implies a false one).}$$

6. "$\equiv$": reads as "equivalent to". It is a binary relation such that, given two formulae (atomic or *wff*): p, q,

$$p \equiv q \text{ is true iff } (p \rightarrow q) \wedge (q \rightarrow p) \text{ is true.}$$

Notice, because of the fact that "$\equiv$" can be defined completely by other connectives, it is therefore reducible to other connectives. The reason we define it separately is based on consideration for its abundant use.

7. "$\neg$": reads as "not". It is a unitary relation such that, given a formula (atomic

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[5]Unlike the common usage in English, the "or" in the language of first order logic is not exclusive.

or *wff*): p,

$$\neg p \text{ is true when } p \text{ is false.}$$

Note that *wff* can be further composed with *wff* or with atomic formulae which would yield to even more complicated *wff*. *Wff*s so defined in the first order language are the standard symbolic formation of sentences in the first order logic. In the following discussion, first order language will be used to describe propositions and theories, or systems of propositions, related to our interests in the number systems of real and hyperreal.

Again, so far we have not yet established the meaning and use of the concept of *language* in first order logic. In order to form meaningful sentences that can be sensibly verified or falsified, language has to be interpreted by a semantic context. Indeed, as analogous to natural languages (by which I mean English, Japanese, or Arabic, etc.), it is the semantic context that really makes it possible for people to translate from one language to another despite the vastly different symbolisms of different languages.

1.2 Structure

Intuitively, structures are the contexts (mostly mathematical for our purpose) where formulae in first order logic language obtain their meaning and consequently truth value. In particular, to emphasise the terminology, a formula is considered as a sentence once it is interpreted by some definite semantic context. To clarify, an interpretation is a one-to-one mapping from a language to a structure. In particular, given a formula in a language, it can be mapped by the interpretation to a sentence with definite meaning and truth value, depending upon the structure. Different structures can have very different interpretations. Analogous to the formation of first order language, a first order structure consists of some individual objects—constants

and variables, and some n-tuple relations. Yet, given the sentence:

$$E\ (\sqrt{4}, 2)$$

(meaning 2 and $\sqrt{4}$ are equal under the standard arithmetic) in a structure that contains ordinary arithmetic, the sentence must be either definitely true or false based on its context (in this case, apparently it is true). In general, different areas of mathematics can be seen as different structures in a logic system. While utilising the same symbolism of logic, each area can lead to different interpretations and yield different truth values given the same formulae in a language. For instance, consider the formula in first order language:

$$R_2(a, b).$$

On one hand, it can be interpreted as $1 < 2$, where R_2 is interpreted as $<$, a and b are interpreted (or assigned) as 1 and 2 respectively, in the structure of natural number system (in which case the sentence is true). On the other hand, it can be interpreted as $\sqrt{4} = 3$, where R_2 is interpreted as $=$, a and b are interpreted (or assigned) as $\sqrt{4}$ and 3 respectively, in the structure of real number system (in which case the sentence is false).

In the rest of the thesis, first order logic that consists of first order language and structure will be employed as the main way to describe the hyperreal number system.

Chapter 2

The Construction Of The

Hyperreal Number System

2.1   The Real Number System $\mathbb{R}$

To begin with, before we delve into the construction of the hyperreal number system $\mathbb{H}$, it is worthwhile to examine the basic features of the real number system $\mathbb{R}$, from which $\mathbb{H}$ is constructed.

While from the perspective of algebraic structure, $\mathbb{R}$ has many specific properties (which will be discussed in further details in chapter 3), for the purpose of constructing $\mathbb{H}$, according to the characterisation by Henle and Kleinberg, $\mathbb{R}$ needs only be conceptualised as a particular structure in the first order logic as described in chapter 1 [5]:

Let $S_{\mathbb{R}} = <r, R_r, F_r>$ be the structure of $\mathbb{R}$, in which it contains:

1. All $r \in \mathbb{R}$;

2. Variables: such as x, y, z that can be used to represent real numbers;

3.   Relations on finite number of real numbers: normally, the relations on $\mathbb{R}$

are either unitary: such as "$I(a)$"—meaning a is an integer,[6] or binary: such as "$R_<(a, b)$"—meaning a is smaller than b.

4. Functions F: $\mathbb{R}^n \to \mathbb{R}^m$. [7] Functions in the structure of $\mathbb{R}$ of course include all the binary operations that constitute the arithmetic on $\mathbb{R}$, such as addition and multiplication. For, addition and multiplication are simply two particular functions on $\mathbb{R}$ where $F_{+/\times}$: $\mathbb{R}^2 \to \mathbb{R}$. Also, functions in the structure of $\mathbb{R}$ include all the ordinary single/multi-variable functions: $F_N : \mathbb{R}^n \to \mathbb{R}$.[8]

2.2   The Hyperreal Number System $\mathbb{H}$

We are now ready to give the definition of $\mathbb{H}$.

Definition 2.1: Given $\mathbb{R}$ as a structure $S_\mathbb{R}$ in first order logic, the hyperreal number system $\mathbb{H}$ is defined to be an extension structure $S_\mathbb{H} = \; < h, R_h, F_h >$ of $S_\mathbb{R}$,[9] which:

1. Contains all real numbers as constants, all relations on $\mathbb{R}$ as relations and all functions on $\mathbb{R}$ as functions (Notice this is essentially the definition of an extension structure in first order logic);

2. Contains special constants that are not contained in $S_\mathbb{R}$ such as infinitesimals and infinities.[10]

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[6]$I(a)$ is such a unitary relation—being an integer—that is used to express membership of the set $\mathbb{Z}$. This is actually how the concept of sets are expressed through first order language.

[7]$\mathbb{R}^n, \mathbb{R}^m$ are referring to Cartesian product spaces based on $\mathbb{R}$. From the point of view of first order logic, These Cartesian products in terms of the concept of *structure* can be defined as: $S_{\mathbb{R}^k} = \; < r^k, R_{r^k}, F_{r^k} >$, where $r^k \in \mathbb{R}^k$, a generic Cartesian product space, R_{r^k}s are relations on $\mathbb{R}^k$, F_{r^k}s are functions whose domains are in $\mathbb{R}^k$.

[8]Functions are indeed covered by the basic machinery of first order language. In effect, functions can be expressed by a combination of relations and implications. For instance, $f(x) = sin(x)$ can be expressed as: $\forall x (R_f(x) \to R_{sin}(x))$, where $R_f(x)$ and $R_{sin}(x)$ are unitary relations that represent the value of each function.

[9]In first order logic, one structure can be an extension of another in the sense that all categories of information in the latter are contained in the former. Here, $S_\mathbb{H}$ being an extension of $S_\mathbb{R}$ is just one example of this relation between structures.

[10]Showing the existence of infinitesimals and infinities is the single most important motivation for

3. Preserves fundamental properties of $\mathbb{R}$. Specifically, properties describable in the same formulae in first order language maintain the same truth value regardless the different interpretations of $\mathbb{R}$ or $\mathbb{H}$. In other words, given a formula in first order language, it is true by the interpretation of $S_{\mathbb{H}}$ *iff* it is true by the interpretation of $S_{\mathbb{R}}$.[11] Thus, properties that are expressible by first order language may be "transferred" between the two structures (this is known as the *Transfer Principle*, which will be discussed in further depth later in this chapter). Notice that this unique relationship between $\mathbb{R}$ and $\mathbb{H}$ makes the discussion of $\mathbb{H}$ practical and intuitive, since it guarantees the sharing of any properties expressible in first order language between $\mathbb{H}$ and $\mathbb{R}$.

While the construction of $\mathbb{H}$ that is introduced in this chapter is based on the concept of *filters* and *ultraproduct*, such concepts need not be and will not be mentioned here directly. Instead, I will provide the precise mathematical ideas underlying the construction of $\mathbb{H}$ in the Appendix. In this chapter, I will trace the construction in a more intuitive and natural way.

To begin with, it would be helpful to first point out that this construction stems from the sequential representation of numbers. Indeed, we can make such representation for real numbers. Recall the decimal representation of real numbers: we know, given a real number r, it can be written as a decimal expansion $n_1.n_2n_3n_4...$ where $n_i \in \mathbb{Z}$ and, for $i \geq 2$, $0 \leq n_i \leq 9$. So, if we separate each digit as an individual integer, then this arbitrarily chosen real number r can be represented as a sequence of integers, namely: $r = n_1, n_2, n_3, n_4...$ Based on this way of representing real numbers, here are some instances: $2 = 2.000... = 2,0,0,0,...$; $3.14 = 3.1400... = 3,1,4,0,0,...$; $\pi = 3, 1, 4, 1, 5, 9, 2, 6...$, etc. Similarly, a hyperreal number h will

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mathematicians to develop and justify $\mathbb{H}$ at the first place.

[11] "*iff*" is the abbreviation of "if and only if", and it will be used through out this thesis.

essentially be constructed as a sequence of real numbers.[12]

Definition 2.2: A hyperreal number is a sequence of reals ordered by natural numbers. Namely,

$$\forall h \in \mathbb{H}, \exists r(1), r(2), r(3), r(4), ..., r(n), ...,$$

where

$$n \in \mathbb{N}$$

and

$$r(n) : \mathbb{N} \to \mathbb{R}$$

such that

$$for\ h\ \in \mathbb{H}, h = (r(1), r(2), r(3), r(4)..., r(n), ...)$$

Here are a few examples of hyperreal numbers from Definition 2.2:

1. All real numbers can be expressed in this form: given $r \in \mathbb{R}$, let r^* be the corresponding element in $\mathbb{H}$. Then

$$r^* = (r, r, r, r, ..)$$

This feature of $\mathbb{H}$ will be used to show that all real numbers are actually contained in $\mathbb{H}$.[13]

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[12]While reals and hyperreals are different in terms of the type of items in their sequential representation–natural numbers for reals but real numbers for hyperreals, they do share one common feature: namely, both of their sequential representations are indexed by natural numbers.

[13]Observe that the way to represent real numbers (including natural numbers) in the hyperreal number system actually differs from that in real number system. For example, the natural number 2 is represented as $(2, 2, 2, ...)$ in $\mathbb{H}$ but as $(2, 0, 0, ..)$ in $\mathbb{R}$.

2. A hyperreal number $h_1 \in \mathbb{H}$ that is not real:

$$h_1 = (1, 2, 4, 8, 16, ..., 2^k, ...)$$

is a *nonstandard* hyperreal that is also an infinity (This will be explained further in details later).

3. Another hyperreal number $h_2 \in \mathbb{H}$ that is not real:

$$h_2 = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ..., \frac{1}{2^k}, ...)$$

is a *nonstandard* hyperreal that is also an infinitesimal (This will be explained further in details later).

To build on this definition of the basic form of hyperreal numbers, perhaps the single most important notion that needs to be introduced and defined is a sense of effectively big subset of natural number $\mathbb{N}$ (this will be formally defined shortly using the notion of *Quasi-Big* sets). To clarify, considering each hyperreal number as a sequence of real numbers indexed by natural number, it is clear that each hyperreal number has a countably infinite *terms* in its sequence. For our purpose of constructing hyperreal number system as a working concept, what matters for a hyperreal number only relies on some effectively large subset of the terms of the entire sequence. In other words, in order to understand the properties of a hyperreal number, we only need to understand the properties of this effectively large collection of terms from the entire sequence that constitutes a hyperreal number.

In order to modify this rudimentary idea into rigorous mathematical standard, the formal definition of this effectively large subset of $\mathbb{N}$ as the standard to measure hyperreal numbers is provided as follow.

Definition 2.3: Quasi-Big set.[14]

Quasi-Big sets are subsets of natural number $\mathbb{N}$ that satisfy the following conditions: given $A, B \subseteq \mathbb{N}$,

1. The cardinality of any Quasi-Big set is infinite;

2. If A and B are Quasi-Big, $A \cap B$ is Quasi-Big;

3. If A is Quasi-Big and $A \subseteq B \subseteq \mathbb{N}$, B is Quasi-Big;

4. Either A or A^c is Quasi-Big;[15]

Before we move on to use this definition of Quasi-Big set, some clarifications of the definition need to be made.

Remarks following **Definition 2.3**.

- From (1) and (4) above in the definition, we know all *cofinite* sets (namely, sets whose complement is finite) must be Quasi-Big. For instance, $\mathbb{N} - \{1, 2, 3\}$ is Quasi-Big, since the set $\{1, 2, 3\}$ is finite.

- Since $A \cap A^c = \emptyset$ and by (1), $\emptyset$ cannot be Quasi-Big, then from (2) we know A and A^c cannot both be Quasi-Big. Therefore, the "or" used in (4) is an exclusive "or". Notice that since the logical connective "$\vee$" in first order logic is inclusive, I will use the symbol "$\underline{\vee}$" to denote exclusive "or"s as such. In addition as a result of the exclusive "or" in (4), when an infinite subset A is not cofinite (meaning both A and A^c are infinite subsets of $\mathbb{N}$), only one of the two can be Quasi-Big. For example, let A be the set of all even natural numbers, then A^c would be the set of all odd natural numbers. Accordingly, only one of them can be Quasi-Big. Notice that the choice between the two can be arbitrary. Yet, once chosen, such choice must be consistent.

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[14]Notice that this name and definition is introduced by James M. Henle and Eugene M. Kleinberg [5].

[15]$A^c = \mathbb{N} - A$.

- The collection of all Quasi-Big sets in $\mathbb{N}$ is an *ultrafilter* of $\mathbb{N}$ (which is a particular subset of the *power set* of $\mathbb{N}$). This will be explained in depth in the appendix.

Now, equipped with the definition of Quasi-Big sets as the adequate measurement of very large sets to compare the terms of hyperreal numbers, we can define relations and functions on $\mathbb{H}$ in order to complete the construction of $\mathbb{H}$ as an extension structure $S_{\mathbb{H}} = <h, R_h, F_h>$ of $S_{\mathbb{R}}$.

Definition 2.4: Relations in $\mathbb{H}$.

Consider some hyperreal numbers: $h_1, h_2, ..., h_n$, where

$$h_k = (r_k(1), r_k(2)...r_k(i)...), \ r_k(i) \in \mathbb{R} \ is \ the \ i^{th} \ term \ of \ h_k.$$

They are in a n-tuple relation $R_n^{\mathbb{H}}(h_1, h_2, ...h_n)$ (or the sentence formed by this relation is true in $\mathbb{H}$) when $\{i \in \mathbb{N} : R_n^{\mathbb{R}}(r_1(i), r_2(i), ...r_n(i))\}$ is Quasi-Big, where: $R_n^{\mathbb{R}}$ is a relation in $\mathbb{R}$ that is in terms of all the ith terms of n hyperreal numbers.

Here are some examples. Consider a binary relation R_b in $\mathbb{H}$. Based on **Definition 2.4**, we know $R_b(h_1, h_2)$ for some $h_1 = (r_1(1), r_1(2), ..., r_1(i), ...)$ and $h_2 = (r_2(1), r_2(2), ..., r_2(i), ...) \in \mathbb{H}$ (meaning that h_1 and h_2 are in this relation R_b), if $\{i \in \mathbb{N} : R_b(r_1(i), r_2(i))\}$ is Quasi-Big. In other words, the relation R_b holds for h_1 and h_2, *iff* it holds for a Quasi-Big collection of corresponding terms of h_1 and h_2.

Equipped with the definition of relation in $\mathbb{H}$, we are ready to demonstrate three especially fundamental binary relations of hyperreal numbers, namely $<, >, \equiv$, through which the order among the individual hyperreal numbers of the structure may be constituted.

Definition 2.5: Ordering relations of $\mathbb{H}$.[16]

For $h_1 = (r_1(1), r_1(2), r_1(3)...)$ and $h_2 = (r_2(1), r_2(2), r_2(3)...) \in \mathbb{H}$,

1. "$\equiv$" is a binary relation "equal" such that: $h_1 \equiv h_2$ iff $\{i \in \mathbb{N} : h_1(i) = h_2(i)\}$ is Quasi-Big;

2. "$<$" is a binary relation "smaller" such that: $h_1 < h_2$ iff $\{i \in \mathbb{N} : h_1(i) < h_2(i)\}$ is Quasi-Big;

3. "$>$" is a binary relation "bigger" such that: $h_1 > h_2$ iff $\{i \in \mathbb{N} : h_1(i) > h_2(i)\}$ is Quasi-Big.

Examples to demonstrate the ordering relations in **Definition 2.4**:

1. Let $h_1 = (1, 1, 1, 1, 1, 1, 1...)$, and $h_2 = (2, 2, 2, 2, 1, 1, 1, 1, 1, ...)$. After a moment of scrutinization, it is easy to see that the real number sequences corresponding to h_1 and h_2 are almost the same everywhere except the first 4 terms. Therefore, S=$\{i \in \mathbb{N} : h_1(i) = h_2(i)\}$ is $\mathbb{N} - \{1, 2, 3, 4\}$ and thus Quasi-Big, since its complement is a finite set $\{1, 2, 3, 4\}$, which makes S a cofinite set (and we know that all cofinite sets are Quasi-Big).

2. Similarly, we can show $(1, 1, 1, 1, 1, 1, 1...) < (1, 1, 1, 1, 2, 2, 2, 2, ..)$ (or $(1, 1, 1, 1, 2, 2, 2, 2, .. > (1, 1, 1, 1, 1, 1, 1...))$.

Next, given three arbitrarily chosen hyperreal numbers:

- $h_1 = (r_1(1), r_1(2), ..., r_1(i), ...)$,

- $h_2 = (r_2(1), r_2(2), ..., r_2(i), ...)$,

- $h_3 = (r_3(1), r_3(2), ..., r_3(i), ...)$.

---

[16]Two notes for the definition of the ordering binary relations: 1) While the standard way to write a binary relation is $R_b(x, y)$, we can alter such representation of these three relations for the purpose of being consistent with our commonsensical way of writing ordering relations. Thus, instead of writing $\equiv (h_1, h_2)$, we would denote it intuitively as $h_1 \equiv h_2$ (similarly for the other two relations). 2) Notice that the "$=$", "$<$", and "$>$" that are used in the definition are referring to the familiar corresponding standard ordering relations in $\mathbb{R}$.

I will demonstrate that the ordering relations so defined in $\mathbb{H}$ can indeed satisfy the standard definition of order relations.

Claim 1: "$\equiv$" in $\mathbb{H}$ is an equivalence relation [9].

Proof.

Step 1. Show Reflexivity: $h_1 \equiv h_1$ (where h_1 is arbitrarily chosen).

Since we know "$=$" in $\mathbb{R}$ is reflexive ($\forall i \in \mathbb{N}\{r_1(i) = r_1(i)\}$), $S = \{i \in \mathbb{N} : r_1(i) = r_1(i)\} = \mathbb{N}$.

$\Rightarrow$ S is Quasi-Big.

$\Rightarrow h_1 \equiv h_1$.

Step 2. Show Symmetry: $h_1 \equiv h_2 \Rightarrow h_2 \equiv h_1$.

Since $h_1 \equiv h_2$, we know $S = \{i \in \mathbb{N} : r_1(i) = r_2(i)\}$ is Quasi-Big.

$\Rightarrow$ Since we know "$=$" in $\mathbb{R}$ is symmetric ($\forall i \in \mathbb{N}\{r_1(i) = r_2(i) \rightarrow r_2(i) = r_1(i)\}$), we know $S' = \{i \in \mathbb{N} : r_2(i) = r_1(i)\} = S$, and thus S' is also Quasi-Big.

$\Rightarrow h_2 \equiv h_1$.

Step 3. Show Transitivity: $(h_1 \equiv h_2) \wedge (h_2 \equiv h_3) \rightarrow (h_1 \equiv h_3)$.

From the premise, we know: $S_{12} = \{i \in \mathbb{N} : r_1(i) = r_2(i)\}$ and $S_{23} = \{i \in \mathbb{N} : r_2(i) = r_3(i)\}$ are Quasi-Big.

$\Rightarrow S_{123} = S_{12} \cap S_{23} = \{i \in \mathbb{N} : (r_1(i) = r_2(i)) \wedge (r_2(i) = r_3(i))\}$ is Quasi-Big (since by definition the intersection of two Quasi-Big sets is Quasi-Big).

Again, since "$=$" in $\mathbb{R}$ is transitive, we know $\forall i \in \mathbb{N}$, $\{(r_1(i) = r_2(i)) \wedge (r_2(i) = r_3(i)) \rightarrow (r_1(i) = r_3(i))\}$.

$\Rightarrow S_{123} \subseteq S_{13} = \{i \in \mathbb{N} : r_1(i) = r_3(i)\}$.

$\Rightarrow S_{13}$ is Quasi-Big, since by definition the super set of a Quasi-Big set is Quasi-Big.

$\Rightarrow h_1 \equiv h_3$.

$\Rightarrow$ From Step 1-3, we can conclude that "$\equiv$" in $\mathbb{H}$ is an equivalence relation. $\square$

Claim 2: "<" and ">" in $\mathbb{H}$ are order relations [9].[17]

Proof (where generic hyperreal numbers h_1, h_2, h_3 defined for above will be used again):

Step 1: Show Comparability: $h_1 \not\equiv h_2 \rightarrow (h_1 < h_2) \veebar (h_2 < h_1)$.

From the premise, we know $S = \{i \in \mathbb{N} : r_1(i) = r_2(i)\}$ is not Quasi-Big.

$\Rightarrow S^c = \{i \in \mathbb{N} : r_1(i) \neq r_2(i)\}$ is Quasi-Big (Note: this is according to the definition of Quasi-Big set).

$\Rightarrow$ Either $\{i \in \mathbb{N} : r_1(i) < r_2(i)\}$ or $\{i \in \mathbb{N} : r_2(i) < r_1(i)\}$ is Quasi Big, since "<" in $\mathbb{R}$ is an order relation.

$\Rightarrow$ Either $h_1 < h_2$ or $h_2 < h_1$.

Step 2: Show Nonreflexivity: $h_1 \not< h_1$.

From Claim 1 we know that $h_1 \equiv h_1$, and thus $S = \{i \in \mathbb{N} : r_1(i) = r_1(i)\}$ is Quasi-Big.

$\Rightarrow \{i \in \mathbb{N} : r_1(i) < r_1(i)\} \subseteq S^c(\{i \in \mathbb{N} : r_1(i) \neq r_1(i)\})$, which is not Quasi-Big.

$\Rightarrow h_1 \not< h_1$.

Step 3: Show Transitivity: $(h_1 < h_2) \wedge (h_2 < h_3) \rightarrow (h_1 < h_3)$.

Notice that the argument to show "<" is transitive is identical to that of "$\equiv$" (similarly, by invoking the transitivity of "<" in $\mathbb{R}$).

From Step 1-3, we can conclude that "<" is indeed an order relation in $\mathbb{H}$.

□

Remarks After Claims 1,2:

1. During the proof process of the two claims, it is evident that the qualification of order relations in $\mathbb{H}$ is founded upon the order relations in $\mathbb{R}$.

2. The existence of these order relations in $\mathbb{H}$ makes $\mathbb{H}$ a *total or linear ordering*

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[17]Here, I will only show "<" is an order relation. A similar argument applies to the case of ">".

space, where:[18]

$$\forall h_1, h_2 \ [(h_1 < h_2) \veebar (h_1 = h_2) \veebar (h_1 > h_2)].$$

3. While the proofs of Claim 1 and 2 are sufficient to show the linear ordering property of $\mathbb{H}$, there is actually more straightforward way to show such property simply by invoking the **Transfer Principle**, which will be clarified further later in this chapter.

Definition 2.6: Functions in $\mathbb{H}$.

A function of n variables in $\mathbb{H}$, namely

$$f_{\mathbb{H}} : \mathbb{H}^n \longrightarrow \mathbb{H},$$

takes in n hyperreal numbers: $h_1, h_2, ...h_n$ and outputs a definite hyperreal number as its value. Namely,

$$f_{\mathbb{H}}(h_1, h_2, ...h_n) = f_{\mathbb{R}}(r_1(1), r_2(1), ...r_n(1)), f_{\mathbb{R}}(r_1(2), r_2(2), ...r_n(2)), ..., f_{\mathbb{R}}(r_1(k), r_2(k), ...r_n(k)), ...$$

where: $f_{\mathbb{R}}$ is the corresponding function in $\mathbb{R}$ such that $f_{\mathbb{R}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $h_k = (r_k(1), r_k(2)...r_k(i)...).$[19]

One thing that needs to be to clarified for the notion of function in $\mathbb{H}$ is the well-definedness. In order to show a function F: $D \longrightarrow R$ to be well-defined, where D is the domain of the function and R is the range of the function, we need to show

$$\forall x, y \in D, \ [x = y \rightarrow f(x) = f(y)].$$

---

[18]The mathematical significance of being a linear ordering space will be clarified further in the next chapter.

[19]Here the value of the function $f_{\mathbb{H}}$ is assumed to be one dimension—a single hyperreal number. However, it need not be so. The value of a function in $\mathbb{H}$ can perfectly be in any finite number dimension similar to the case of $\mathbb{R}$. Thus, a more general form of function in $\mathbb{H}$ would look like: $f_{\mathbb{H}}$: $\mathbb{H}^n \longrightarrow \mathbb{H}^m$ where n and m are some natural numbers.

Remarks concerning **Definition 2.6**: Functions in $\mathbb{H}$ are well-defined.

Proof.

To begin with, notice that I will only supply a proof for the case of one-place function F: $\mathbb{H} \longrightarrow \mathbb{H}$. Similar arguments can be made for more complicated functions.

Given $h_1 = (r_1(1), r_1(2)...r_1(i)...)$ and $h_2 = (r_2(1), r_2(2)...r_2(i)...) \in \mathbb{H}$, let $h_1 \equiv h_2$.

To prove the well-definedness of a function, we need to show $f_{\mathbb{H}}(h_1) \equiv f_{\mathbb{H}}(h_2)$.

From the equivalence relation between h_1 and h_2, we know $S = \{i \in \mathbb{N} : r_1(i) = r_2(i)\}$ is Quasi-Big.

Since we know functions are well-defined in $\mathbb{R}$, $\forall i \in \mathbb{N}$, $[r_1(i) = r_2(i) \rightarrow f_{\mathbb{R}}(r_1(i)) = f_{\mathbb{R}}(r_2(i))]$.

Thus, $\forall i \in S$, $[f_{\mathbb{R}}(r_1(i)) = f_{\mathbb{R}}(r_2(i))]$.

Thus, S $\subseteq \{i \in \mathbb{N} : f_{\mathbb{R}}(r_1(i)) = f_{\mathbb{R}}(r_2(i))\}$.

Thus, $\{i \in \mathbb{N} : f_{\mathbb{R}}(r_1(i)) = f_{\mathbb{R}}(r_2(i))\}$ is Quasi-Big (according to the definition of Quasi-Big set).

Thus, $f_{\mathbb{H}}(h_1) \equiv f_{\mathbb{H}}(h_2)$.

In conclusion, functions in $\mathbb{H}$ are well-defined. $\square$

To finish the basic construction of $\mathbb{H}$, I will provide some particularly important examples of functions—*Arithmetic operations of $\mathbb{H}$ in terms of some 2-place functions*— in $\mathbb{H}$ that will be relevant for the discussion in the next chapter.

Definition 2.7: Basic Arithmetic Operations on $\mathbb{H}$.

Given two generic hyperreal numbers

$$h_1 = (r_1(1), r_1(2)...r_1(i)...) \text{ and } h_2 = (r_2(1), r_2(2)...r_2(i)...) \in \mathbb{H},$$

arithmetic operations "+" (Addition) and "×" (Multiplication) are binary operations

(or 2-place functions) on $\mathbb{H}$ such that:

1. $h_1 + h_2 = (r_1(1) + r_2(1), r_1(2) + r_2(2), r_1(3) + r_2(3), ..., r_1(i) + r_2(i), ...)$ (adding term by term).

2. $h_1 \times h_2 = (r_1(1) \times r_2(1), r_1(2) \times r_2(2), r_1(3) \times r_2(3), ..., r_1(i) \times r_2(i), ...)$ (multiplying term by term).

2.3 Justifications of the Hyperreal Number System

Now we can proceed to show the hyperreal number structure $S_{\mathbb{H}} = <h, R_h, F_h>$ so constructed in this chapter does indeed satisfy all the requirements in **Definition 2.1**.

1. $S_{\mathbb{H}} = <h, R_h, F_h>$ is an extension on $S_{\mathbb{R}} = <r, R_r, F_r>$.[20]

This can be proven by first showing all real numbers are contained in $\mathbb{H}$ as we constructed it: given any $r \in \mathbb{R}$, r corresponds to $(r, r, r, ..., r, ...) \in \mathbb{H}$. Since this mapping clearly is injective, we know all real numbers are distinctively represented in $\mathbb{H}$. Thus, all definable relations and functions in $\mathbb{R}$ are also contained in $\mathbb{H}$, for, given any function/relation in $\mathbb{R}$, we can define a corresponding function/relation in $\mathbb{H}$ simply by limiting the domain of the function/relation to $\mathbb{R} \subseteq \mathbb{H}$. Therefore, $\mathbb{H}$ as a structure fully contains $\mathbb{R}$ as its substructure.

2. $\mathbb{H}$ contains "new" elements: Infinitesimals and Infinities.

To show this, let's start with showing the existence of some infinities in $\mathbb{H}$.

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[20]Or alternatively: $S_{\mathbb{R}}$ is contained in $S_{\mathbb{H}}$.

Claim: $\omega = (1, 2, 3, ..., i, ...)$, for $i \in \mathbb{N}$, is an infinity in $\mathbb{H}$; i.e. $\forall r \in \mathbb{R}, \ \omega > r$.

Proof. To prove this, we need to invoke a special property of $\mathbb{R}$, namely, the *Eudoxus-Archimedes Principle* [4]:[21]

$$\forall r \in \mathbb{R}, \exists n \in \mathbb{N}, \ [r < n].$$

Accordingly, we know

$$S = \{i \in \mathbb{N} : i < r\} \subseteq \{i \in \mathbb{N} : i < n\}$$

$\Rightarrow$ S is finite.

Therefore, we know $S^c = \{i \in \mathbb{N} : i > r\}$ is cofinite, and thus **Quasi** − **Big**.

Therefore based on the definition of order relations in $\mathbb{H}$, we can conclude that:

$$\forall r \in \mathbb{R}, \ [\omega > r].$$

Hence, ω is an infinity.

$\square$

Similarly, we can construct an infinitesimal in $\mathbb{H}$. Let $\epsilon = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{i}, ...)$. It can be proven to be an infinitesimal by invoking an alternative version of *Eudoxus-Archimedes Principle*: namely, $\forall r \in \mathbb{R}, \exists n \in \mathbb{N}, \ [\frac{1}{n} < r]$ (while the example of infinitesimal and infinity provided here are both positive, they need not to be so).[22]

---

[21]Notice that this principle can be formulated in first order language, and therefore, according to the Transfer Principle, the corresponding form of it should hold as well in $\mathbb{H}$ (This will be discussed further in the next section).

[22]Notice that once we accept the Transfer Principle, an alternative way to prove the existence of infinitesimals can be induced as follow. Given an infinity ω, we know on the one hand $\omega \neq 0$, and

27

3. The Transfer Principle holds: Any sentence expressible in first order language is true in $\mathbb{H}$ *iff* it is true in $\mathbb{R}$.

To prove that the hyperreal number system constructed as such indeed suffices its definition, the last step and probably also the key step, is to prove the Transfer Principle. Due to its tremendous importance, let's first examine further why this seemingly extravagant principle is actually quite plausible by looking at some well-known examples. Although I will provide independent proofs of these examples without using the Transfer Principle, these proofs will be deemed as unnecessary once we accept the Transfer Principle. While it might seem to be a little detour by giving unnecessary proofs, such efforts perhaps would help to demonstrate the extraordinary power of the Transfer Principle. After examining the examples, I will present a provisional proof for the principle itself.

2.4 The Transfer Principle: Use/Misuse and Applicability/Non-applicability

Case 1: *Eudoxus-Archimedes Principle* [4].

As I have invoked in the proof of the existence of an infinity in $\mathbb{H}$, we know back in $\mathbb{R}$, given any real number, there always exists a natural number that is bigger than the real number. Again, to put it in the form of first order language,

$$\forall r \in \mathbb{R}, \exists n \in \mathbb{N}, \ [n > r].$$

therefore $\dfrac{1}{\omega} \neq 0$ (due to the Transfer Principle). On the other hand, we know $\frac{1}{\omega} \in \mathbb{H}, \forall r \in \mathbb{R}, \ [\frac{1}{\omega} < r]$ (which is also due to the Transfer Principle). Thus, $\dfrac{1}{\omega}$ is by definition an infinitesimal.

In addition, this property can be put differently in the sense that there does not exist a real number that is bigger than all natural numbers. In particular, this can also be expressed as:

$$\nexists r \in \mathbb{R}, \ \forall n \in \mathbb{N}, \ [\underbrace{1+1+1+...+1}_{\text{n number of 1s}} < r].$$

This is to say: no matter how big the number is in $\mathbb{R}$, it can always be reached by adding a sufficient number of 1s. In fact, this feature can be generalised to any two numbers in $\mathbb{R}$, and thus $\mathbb{R}$ is considered as an *Archimedean Group*.

However, such is not the case for $\mathbb{H}$: in fact, I will use the following claim to provide a counterexample to show $\mathbb{H}$ is actually not Archimedean.

Claim: "$\forall h \in \mathbb{H}, \exists n \in \mathbb{N}, \ [n > h]$" is **false**!

Proof. This proof follows the proof of the existence of infinity in $\mathbb{H}$.

Simply let ω be an infinity in $\mathbb{H}$. We know $\forall r \in \mathbb{R}, \omega > r$. Since $\mathbb{N} \subseteq \mathbb{R}, \forall n \in \mathbb{N}, \ [n \in \mathbb{R}]$.

Therefore, $\forall n \in \mathbb{N}, \ [\omega > n]$.

Therefore, $\exists h \in \mathbb{H}$, one instance of which is ω, $\nexists n \in \mathbb{N}, \ [n > h]$.

$\square$

As a result of the proof of this claim, it can also be claimed that there exists some hyperreals, no matter how many finite number of 1s are added, the sum is always smaller than these hyperreals. In other words, the defining criterion of $\mathbb{H}$ being a *Archimedean-group*, namely,

$$\nexists h \in \mathbb{H}, \ \forall n \in \mathbb{N}, \ [\underbrace{1+1+1+...+1}_{\text{n number of 1s}} < h], \quad \textit{is false!}$$

While $\mathbb{H}$ and $\mathbb{R}$ are at odds in terms of being Archimedean, such different behaviour does not show the failure of the Transfer Principle. According to the Transfer Principle, we know *Eudoxus-Archimedes Principle* should also hold in $\mathbb{H}$ (Since it can be formulated in first order language). For, in the corresponding sentence of the *Eudoxus-Archimedes Principle* in $\mathbb{H}$, the object of predication should not the standard $\mathbb{N} \subseteq \mathbb{R}$ anymore; rather, it should be transferred to the *hypernatural* numbers, denoted as $\mathbb{N}^*$.[23] Consequently, whereas $\mathbb{H}$ fails to be Archimedean as it is shown, it could at the same time satisfies the *Eudoxus-Archimedes Principle*, by satisfying the following claim.

Claim: $\forall h \in \mathbb{H}, \ \exists n^* \in \mathbb{N}^*, \ [n^* > h]$.

Proof. For $a_i \in \mathbb{R}$, let $h = (a_1, a_2, a_3, ..., a_i, ...)$ be a generic hyperreal number. Since we know the *Eudoxus-Archimedes Principle* holds in $\mathbb{R}$, therefore

$$\forall a_i, \ \exists n_i \in \mathbb{N}, \ [n_i > a_i].$$

Thus, we can construct the hypernatural number $n^* = (n_1, n_2, n_3, ..., n_i, ...)$ where $n_i > a_i$.

Clearly $\{i \in \mathbb{N} : n_i > a_i\}$ is the whole $\mathbb{N}$, Since $n_i > a_i$ everywhere!

Therefore, $\{i \in \mathbb{N} : n_i > a_i\}$ is Quasi-Big.

Therefore, $n^* > h$.

Therefore, $\forall h \in \mathbb{H}, \ \exists n^* \in \mathbb{N}^*, \ [n^* > h]$!

$\square$

---

[23]Notice that many misuses of the Transfer Principle happen in the similar manner: when the object of the predication is left un-transferred. Indeed, it is based on the transfer of sets that the Transfer Principle becomes viable through model theory. In other words, to correctly describe the corresponding transferred property, one has to "internalize" the objects of predication (This also leads to the development of *Internal Set* [10] [4]).

Case 2: *Some properties concerning infinitesimals.*

For our purpose that will be presented in the next chapter, I need to further clarify some additional properties of infinitesimals through the following theorem (the proof of which relies on the Transfer Principle).

Theorem 2.1 [5]

Suppose $\delta_1, \delta_2 \in \mathbb{H}$ are positive infinitesimals, and $r \in \mathbb{R}^+$ is a positive real (notice this theorem is exclusively focusing on the positive hyperreal numbers to avoid any unnecessary complication of the problem, yet similar properties hold as well for the negative or the entire hyperreal number set).

- $\delta_1 \cdot r$ is an infinitesimal;

- $\delta_1 \cdot \delta_2$ is an infinitesimal;

- $\delta_1 + \delta_2$ is an infinitesimal.

Proof. [24]

To prove a number is infinitesimal, by definition we need to show the number is nonzero (in this case positive) and its absolute value (in this case the number itself) is smaller than all positive real numbers.

1. Show $\delta_1 \cdot r$, $\delta_1 \cdot \delta_2$, and $\delta_1 + \delta_2$ are nonzero.

Recall in $\mathbb{R}$ we know $\forall a, b \neq 0[(a \cdot b \neq 0) \wedge (a + b \neq 0)]$. By the Transfer Principle, we know so this is also the case in $\mathbb{H}$. Therefore, since $\delta_1, \delta_2, r \neq 0$, we can conclude that $\delta_1 \cdot r$, $\delta_1 \cdot \delta_2$, and $\delta_1 + \delta_2$ are also nonzero.

2. Show: $\forall r' \in \mathbb{R}^+$, $\delta_1 \cdot r$, $\delta_1 \cdot \delta_2$, and $\delta_1 + \delta_2 < r'$.

Since $\delta_1, \delta_2 \in \mathbb{H}$ are some positive infinitesimals, we know $\forall r' \in \mathbb{R}^+$ it is the case that:

1. $\delta_1 < r'/r$,

---

[24]the proof provided here is due to Henle and Kleinberg [5].

2. $\delta_1 < r'$,

3. $\delta_2 < 1$,

4. and $\delta_1, \delta_2 < r'/2$.

From 1, we can see $\delta_1 \cdot r < (r'/r) \cdot r$, and thus $\delta_1 \cdot r < r'$ (notice that this and what follows are due to the Transfer Principle, since they are all formulated in first order language, and the corresponding sentences in $\mathbb{R}$ are all true).

From 2,3, we may see that $\delta_1 \cdot \delta_2 < r'$.

From 4, we can see $(\delta_1 + \delta_2) < (r'/2 + r'/2)$, and thus $\delta_1 + \delta_2 < r'$.

In conclusion, we know $\delta_1 \cdot r$, $\delta_1 \cdot \delta_2$, and $\delta_1 + \delta_2$ are all infinitesimals!

$\square$

Corollary 2.1.1: There exists an uncountably infinite number of infinitesimals.[25]

Proof. From **Theorem 2.1**, we know given any infinitesimal δ and nonzero standard hyperreal number r, $\delta \cdot r$ is also infinitesimal. Accordingly, let r_1, r_2 be two generic instances of such nonzero standard hyperreal numbers. We know $\delta \cdot r_1$ and $\delta \cdot r_2$ are infinitesimals. Claim: $\delta \cdot r_1 \neq \delta \cdot r_2$, if $r_1 \neq r_2$. to justify, we need to show that $\delta \cdot r_1 - \delta \cdot r_2 \neq 0$ or $\delta \cdot (r_1 - r_2) \neq 0$. This is easy to show: since we know $\delta \neq 0$ and $r_1 \neq r_2$, then $\delta \cdot r_1 - \delta \cdot r_2 = \delta \cdot (r_1 - r_2) \neq 0$ because of the Transfer Principle. Therefore, we know $\delta \cdot r_1$ and $\delta \cdot r_2$ are two distinct infinitesimals when $r_1 \neq r_2$. Recall it is known that there are uncountably many nonzero reals, thus there should be uncountably many infinitesimals as a result. Observe that what I have shown here is essentially that there is an injective mapping from $\mathbb{R} - \{0\}$ (which has the same cardinality as $\mathbb{R}$) to $\mathbb{I}$, where $\mathbb{I}$ is the set of all infinitesimals. $\square$

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[25]To Marie Tolan's credit, the proof of this corollary also amounts to proving that *there exists an uncountably infinite number of infinities*! Indeed, since there exists an injective mapping between infinitesimals to infinities, i.e.by taking the reciprocal of each infinitesimal, there must be at least as many infinities as infinitesimals (observe that taking the reciprocal is indeed an injective mapping due to the Transfer Principle).

Case 3: *The Least Upperbound Property.*

As one of the fundamental properties of $\mathbb{R}$, it is also commonly known as the *Axiom of Completeness*. It could be formulated as: for all nonempty sets of numbers (or real numbers in this case), if there is an upper-bound, then there must be a least upper-bound. This property can be used to characterise the completeness of the space of $\mathbb{R}$. While intuitively speaking, it seems that since hyperreals are a lot more "densely" populated than reals (this will be further clarified in the next chapter), it would be plausible to speculate that $\mathbb{H}$ should be also complete in this sense of the *least upper-bound property.* However, if we scrutinise the symbolic representation of this property, it is not hard to see that this is surprisingly not the case. In other words, even though hyperreals are seemingly much more "densely" populated,[26] $\mathbb{H}$ does not satisfy the *Least Upper Bound Property*: i.e. $\mathbb{H}$ is not a complete space!

To see why this is the case, we can start by looking at the expression of this property. In general, given $\mathbb{S}$ as a *nonempty* set of numbers of the universe of a number system $\mathbb{U}$, and $\mathbb{M} \subseteq \mathbb{U}$ as the set of all potential upper bounds of $\mathbb{S}$, the *Least Upper Bound Property* can be formulated as:

$$\forall \mathbb{S} \subseteq \mathbb{U}, \ \{[\exists m \in \mathbb{M}, \forall s \in \mathbb{S}, \ (s \leq m)] \to [\exists m_{min} \in \mathbb{M}, \forall m \in \mathbb{M}, \ (m_{min} \leq m)]\}.$$

Notice that one unique feature of this expression is that the quantifier "$\forall$" at the beginning of the expression is quantifying over sets (or relations syntactically speaking) of numbers, i.e. $\mathbb{S}$. Since, recall in first order language, the only legitimate category of objects of quantifiers are individual variables, the expression of the *Least Upper Bound Property* is hence beyond the machinery of the first order language! As

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[26]In fact, around each real number there is an interval known as the *halo* or *monad* of the real number defined by some nonstandard hyperreal numbers within which there is no any other real number but infinitely many hyperreals! Note: this will be clarified further in the next chapter.

a matter of fact, this also points to the difference between the first and second order languages. While first order sentences can only predicate/quantify over individual variables, second order sentences, in addition, allow predication/quantification over relations (which include all the sets as some unitary relations) [11]. As a consequence of the limitation of the first oder language, the *Axiom of Completeness* (or the *Least Upper Bound Property* cannot be transferred from $\mathbb{R}$ to $\mathbb{H}$. Furthermore, although the fact that the *Axiom of Completeness* cannot be proven by the Transfer Principle does not automatically disprove it, this property actually can be proven to be false in $\mathbb{H}$ with an easy counterexample.

Claim: $\mathbb{H}$ does not satisfy *The Least Upper Bound Property* [5].

Proof by Counterexample. Consider the set F of all *finite* hyperreals, i.e. hyperreals that are smaller than some natural number or real number but bigger than some other natural number or real number.

To begin with, notice that:

- F contains at least all the reals, since each real number is finite as a result of the *Eudoxus-Archimedes Principle.* Therefore, F must not be empty.

- F has an upper bound: $h = (1, 2, 3, ..., n, ...)$ (or any infinity in $\mathbb{H}$).

However, it is claimed that F does not have a least upper bound.

Assume $\exists$ a least upper bound $w \in \mathbb{H}$.

$\Rightarrow w$ is an upper-bound;

$\Rightarrow$ by the definition of upper-bound, we know $\forall s \in F,\ [w > s]$;

$\Rightarrow w$ is an infinity (since it is bigger than all finite hyperreals);

$\Rightarrow w - 1$ is also an infinity in $\mathbb{H}$;[27]

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[27]While this seems plausibly true, I will still give a rigorous proof for it in a latter section, where I specifically discuss the properties of infinity in $\mathbb{H}$.

$\Rightarrow w - 1$ is another upper-bound and $w - 1 < w$.

$\Rightarrow w$ is not the least upper-bound after all! $\Rightarrow$ Contradiction!

Therefore, there does not exist a least upper-bound for the set of all finite hyperreals F.

Therefore, it is not the case that all nonempty sets of hyperreals that have an upper-bound would have a least upper-bound. $\square$

Now, let's turn to the actual proof of the Transfer Principle [5]:

To begin with, suppose a generic first order sentence G in $\mathbb{H}$ that predicates $n \in \mathbb{N}$ hyperreal numbers. Namely, $G(h_1, h_2, h_3, ..., h_n)$, where $h_k \in \mathbb{H}$ such that

$$h_k = (r_k(1), r_k(2), r_k(3), ..., r_k(i), ...).$$

Observe that a corresponding sentence in $\mathbb{R}$ that can be derived from G is:

$$G_i(r_1(i), r_2(i), r_3(i), ..., r_k(i), ...),$$

where $r_k(i) \in \mathbb{R}$ is the ith term in h_k (Recall each hyperreal number is just a sequence of real numbers indexed by natural number). Notice that G_n therefore is essentially a sentence predicating over all the ith terms from the n hyperreal numbers.

Based on this set-up, we can introduce the following key theorem in proving the Transfer Principle by connecting G and G_n.

Theorem 2.2 (*Łos' Theorem*):[28]

Given G defined as above, G is true in $\mathbb{H}$ *iff* $I = \{i \in \mathbb{N} : G_i$ is true in $\mathbb{R}\}$ is

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[28]This theorem first proven by Jerzy Łos is also known as the *Fundamental Theorem of Ultraproducts.*

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Quasi-Big.

Despite the tremendous importance of this theorem, the proof of it should be at least intuitively sensible. For, most importantly, recall back in **Definition 2.4**, we have defined the truth value of each relation in $\mathbb{H}$ in the same way. To clarify, since G is a sentence that is formulated in the first order language, no matter how complicated the sentence is, it must boil down to some individual relations that are connected by some connectives. Since the truth value of each component relation, denoted as R_j, in G depends only on a Quasi-Big set of real terms of the hyperreal numbers that are in R_j, it can be proven that the truth value of the overarching sentence G may also be determined only by a Quasi-Big set of real terms of the n hyperreal numbers. Indeed, the complete proof, which will not be provided here due to its tediousness, precisely relies on this basic intuition. Specifically, to prove **Theorem 2.2**, one can break down all the possible ways to construct the sentence G in the first order language, and then examine each category of them.[29]

A provisional proof of the Transfer Principle is provided as follow. Given G as a sentence in both $\mathbb{R}$ and $\mathbb{H}$, if G is true in $\mathbb{R}$, then G_i is true for all $i \in \mathbb{N}$. Therefore, G_i is true for a Quasi-Big set of terms of the hyperreals that G contains. Hence, G must be true in $\mathbb{H}$ as well. Alternatively, if G is false in $\mathbb{R}$, then G_i is false for all $i \in \mathbb{N}$. Therefore, G_i is false for a Quasi-Big set of terms of the hyperreals that G contains. Hence, G must be false in $\mathbb{H}$ as well. In conclusion, G is true in $\mathbb{H}$ *iff* it is true in $\mathbb{R}$.

At this point, based on the prolonged efforts from Section 2.3 and Section 2.4, we can conclude that $\mathbb{H}$, which is constructed in Section 2.2, is indeed a construction that suffices all the requirements in the definition of the hyperreal number system: namely, it fully contains the real number system; it contains infinities and infinitesimals; and

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[29]A more detailed proof of such an induction can be found in the Appendix B of [5].

the Transfer Principle holds between $\mathbb{R}$ and $\mathbb{H}$.

Chapter 3

Investigations Of Some Important Properties Of The Hyperreal Number System

In this chapter, based on the construction and proof of the functionality of $\mathbb{H}$, I will further provide accounts for the properties of this nonstandard system, which hopefully will bring in more understanding of $\mathbb{H}$. In particular, while it is a broad area with enormous amount of information, here I will focus on some most basic and fundamental ones, especially by considering the areas of *topology, set theory, and abstract algebra.*

3.1   Basic Categorisations of Hyperreals

I have not yet formally defined concepts such as *hypernaturals* (the "natural numbers" in $\mathbb{H}$), so, to begin with, I will give the explicit definition along with some other basic category of hyperreal numbers.

Definition 3.1.1:

1. $h = (r_1, r_2, r_3, ..., r_i, ...) \in \mathbb{H}$ is a **hypernatural** number when the set $\{r_i : r_i \in \mathbb{N}\}$ is Quasi-Big. The set of all hypernatural numbers is denoted as $\mathbb{N}^*$.

2. $h = (r_1, r_2, r_3, ..., r_i, ...) \in \mathbb{H}$ is a **hyperinteger** when the set $\{r_i : r_i \in \mathbb{Z}\}$ is Quasi-Big. The set of all hyperinteger numbers is denoted as $\mathbb{Z}^*$.

3. $h = (r_1, r_2, r_3, ..., r_i, ...) \in \mathbb{H}$ is a **hyperrational** number when the set $\{r_i : r_i \in \mathbb{Q}\}$ is Quasi-Big. The set of all hyperrational numbers is denoted as $\mathbb{Q}^*$.

Remarks concerning **Definition 3.1.1**

Notice that in the definition of subcategories of hyperreals, the determining factor of a hyperreal is a Quasi-Big set of its real terms.

Before moving on to discuss the properties of $\mathbb{H}$, one more set of definitions needs to be given as follow.

Definition 3.1.2:

Given a hyperreal number h,

1. h is limited if

$$\exists r_1, r_2 \in \mathbb{R}, \ such \ that \ r_1 < h < r_2.$$

Otherwise, h is unlimited, or when

$$\forall r \in \mathbb{R}, \ [(h > r) \vee (h < r)].$$

2. h is standard if $h \in \mathbb{R}$; otherwise h is nonstandard (when $h \in (\mathbb{H} - \mathbb{R})$).

Remarks concerning **Definition 3.1.2**

Notice that while the notions of limited and unlimited hyperreals are essentially referring to the "size" of the number,[30] the notions of standard and nonstandard hyperreals are aiming at distinguish the "new" numbers in $\mathbb{H}$ from the ones already existing in $\mathbb{R}$, which is similar to the distinction between rational numbers and irrational numbers in $\mathbb{R}$.

3.2 Topological Properties of Hyperreals

Recall, back in $\mathbb{R}$, we know any two elements are comparable in the sense that for any two real numbers, one must be bigger, smaller, or equal to the other (notice that the "or" here is exclusive). Such property of $\mathbb{R}$ is known as a total (or linear) ordering [6]. As Henle and Kleinberg point out, one common way to express this feature visually is to say: the space that contains all $r \in \mathbb{R}$ can be put into a single line (or in a one-dimensional space), which in turn gives birth to the notion of the well-known *real number line*. It is worth pointing out that this total ordering property can actually be boiled down to two sentences in the first order language [5]:

1. $\forall x, y, \ \{(x \neq y) \rightarrow [(x < y \vee x > y) \wedge \neg(x < y \wedge y < x)]\}$.[31]

2. $\forall x, y, z, \ [(x < y \wedge y < z) \rightarrow (x < z)]$.

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[30]It worth mentioning that "limited" and "unlimited" are in fact an alternative pair of namings to "finite" and "infinite" (namely, "limited" is equivalent to "finite", and "unlimited" is equivalent to "infinite"). The rationale for making such repetitive definitions, according to [4], is that the new definition/naming can avoid potential misleading implications as the old one would lead to. To explain, consider an unlimited hypernatural. While technically it is infinite, it nevertheless maintains many properties of natural numbers (which are finite) due to the Transfer Principle. In fact, these unlimited hypernatural numbers are also known as *transfinite* natural numbers to avoid the disjunctive connotations between "finite" and "infinite".

[31]Notice that the "=" here would be referring to the equivalence relation "$\equiv$" defined in Chapter 2 for the case of $\mathbb{H}$.

While the first sentence describes the exclusiveness of the "or" relation in the ordering, the second sentence describes the transitivity of the ordering. In any case, the fact that both of the sentences can be put into first order language makes the total ordering property a subject of the Transfer Principle. Thus, we can conclude that $\mathbb{H}$ must also have the total ordering property. In other words, the space that is formed by all $h \in \mathbb{H}$ is also one dimensional, and we call it the *Hyperreal Number Line*.

Even though both $\mathbb{H}$ and $\mathbb{R}$ qualify as linear spaces as a result of the total ordering property, the corresponding number lines, however, are very different in nature. We will start the further examination of the topology of $\mathbb{H}$ and $\mathbb{R}$ by giving the necessary definitions of some new relations in $\mathbb{H}$ which are widely used in the field [4].

Definition 3.2.1[32]

Given two hyperreal numbers h_1, h_2,

- h_1 is *infinitely close* to h_2, denoted as $h_1 \cong h_2$, if $(h_1 - h_2)$ is infinitesimal or 0. Furthermore, for any hyperreal h, we define the *halo* of h: $hal(h) = \{h_i \in \mathbb{H} : h_i \cong h\}$[33].

- h_1 is of *limited distance* to h_2, denoted as $h_1 \simeq h_2$, if $(h_1 - h_2)$ is limited. Furthermore, for any hyerperreal h, we define the *galaxy* of h: $gal(h) = \{h_i \in \mathbb{H} : h_i \simeq h\}$.

Remarks concerning **Definition 3.2.1**

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[32]The arithmetic operation "$-$" is defined similarly to those that are defined in chapter 2.

[33]Historically, the halo of a hyperreal number h is also known as the *monad* of h which is named by Abraham Robinson in honour of the metaphysical notion of "monad" supposedly coined by Gottfried Leibniz [10].

Using the technique similar to that of proving "≡" as an equivalence relation in $\mathbb{H}$, we may show that both "≅" and "≃" are also equivalence relations in $\mathbb{H}$! Thus, given $h \in \mathbb{H}$, $hal(h)$ can be seen as the "≅-equivalence" class of h and gal(h) can be seen as the "≃-equivalence" class of h (in addition to the normal " ≡-equivalence class" of h, the members of which are considered as strictly equal to h). In fact, the ≡-equivalence is a stricter form of ≅-equivalence in the sense that $h_1 \equiv h_2$ implies $h_1 \cong h_2$ (since $h_1 - h_2$ is strictly 0 when $h_1 \equiv h_2$). Similarly, we can see that ≅-equivalence is a stricter form of ≃-equivalence in the sense that $h_1 \cong h_2$ implies $h_1 \simeq h_2$ (since 0 or infinitesimals are definitely limited hyperreals).

As a consequence of these additional equivalence relations in $\mathbb{H}$, we may derive further insights about the hyperreal number line. While we know that, because of the nature of equivalence relation in general, all three equivalence relations: $\equiv$, $\cong$, and $\simeq$ may introduce a particular partition of the hyperreal number space, the partitions introduced respectively by $\cong$ and $\simeq$ may tell us more about the topology of $\mathbb{H}$.

First, consider ≅-equivalence. For a generic ≅-equivalence class $hal(h)$, it contains all the hyperreals that are infinitely close to h. Assume x is such a generic hyeperreal in the halo of h. To be infinitely close to h in turn requires x either to be identical to h itself (in which case $x \equiv h$) or to be only in an infinitesimal "distance" to h (in which case $x - h$ is some infinitesimal). Such feature of the notion of halo becomes more illuminating when $h \in \mathbb{R}$. To clarify, let's suppose h is a standard hyperreal (i.e. $h \in \mathbb{R}$). Since we know that all infinitesimals are nonstandard, we can show that x must be nonstandard provided x is different from h. Note that this is a direct result from the fact that the sum of any standard hyperreal and any nonstandard hyperreal is nonstandard.[34] The fact that any element of the halo of a standard hyperreal h is nonstandard is significant: it amounts to saying that all standard hyperreals

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[34]The proof of this is fairly straightforward and thus will not be provided here. For those who are interested, a complete proof can be found on Page 35 in [5].

are contained in some distinctive interval of nonstandard hyperreals that completely isolate every standard hyperreal from other standard hyperreals. In other words,

$$\forall h_r \in \mathbb{R}, \nexists h_r' \in \mathbb{R}, \ [h_r' \in hal(h)].$$

Such a feature of the halo of any standard hyperreals in $\mathbb{H}$ shows one important implication: that the hyperreal number line is much "denser" than the real number line. As we have proven in Corollary 2.1.1, there are an uncountably infinite number of infinitesimals, and it is easy to see that $hal(h_r)$ also contains uncountably many nonstandard elements. For, given any infinitesimal δ, we know $h \pm \delta \in hal(h)$. This demonstrates that for any two standard hyperreals, no matter how close they are (from our experiences and the completeness of $\mathbb{R}$, they should be "really close"), they are always disjointed by uncountably infinite number of nonstandard hyperreals! In fact, I will prove the following theorem to show that $\mathbb{R}$ is actually not *dense* in $\mathbb{H}$!

Theorem 3.2.1:[35] $\mathbb{R}$ is *not dense* in $\mathbb{H}$ in the sense that: there exists an open set in $\mathbb{H}$ that does not contain any reals (or standard hyperreals).

Proof. Here is an easy example: let δ be a positive infinitesimal, and $S \subseteq \mathbb{H} = \{s \in \mathbb{H} : \delta/3 < s < \delta\}$. Notice that S is nonempty, for we know at least the hyperreal number $\delta/2 \in S$ (note that again this is due to the Transfer Principle); also notice that S is considered as an open set in $\mathbb{H}$, since it is an open interval in $\mathbb{H}$ (which is also a result of the Transfer Principle). Clearly, we can see that

$$\nexists r \in \mathbb{R}, \ [\delta/3 < r < \delta].$$

Hence, $\forall r \in \mathbb{R}[r \notin S]$.

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[35]Notice that the proof provided here is completely derived from the discussion between Professor Bloch and me.

Therefore, we conclude that $\mathbb{R}$ is *not dense* in $\mathbb{H}$!

$\square$

Next, let's consider $\simeq$-equivalence. Given a $\simeq$-equivalence class $gal(h)$, it contains all the hyperreals that are at a limited "distance" to h. Assume x is such a generic hyperreal in the galaxy of h. Again, analogously to the case of a halo, it will help us understand the topology of $\mathbb{H}$ if we assume h is standard (i.e. $h \in \mathbb{R}$). Thus, let's assume h is standard. By definition, we know

$$gal(h) = \{x \in \mathbb{H} : r_1 < x - h < r_2, \text{ for some } r_1, r_2 \in \mathbb{R}\}.$$

Since $h \in \mathbb{R}$, we know

$$gal(h) = \{x \in \mathbb{H} : r_1 + h < x < r_2 + h\} = \{x \in \mathbb{H} : r_1' < x < r_2', \text{ for some } r_1', r_2' \in \mathbb{R}\}.$$

Notice that as we know that $\forall r \in \mathbb{R}, [r_1' < r < r_2']$, we conclude that $\mathbb{R} \subseteq gal(h)$ when h is standard. However, there exists some nonstandard hyperreal numbers that are *not* in the galaxy of a standard h.

Indeed, a simple category example of such nonstandard hyperreals would be any unlimited hyperreal number (any infinities in $\mathbb{H}$). Let's consider an unlimited hyperreal ω, i.e. $\forall r \in \mathbb{R}, [r < \omega]$. Since the difference between ω and h, $\omega - h$, is also an unlimited hyperreal, we can conclude that ω is not in the galaxy of h. The existence of unlimited elements in $\mathbb{H}$ that are outside of the galaxy of h indicates that not only is the hyperreal number line categorically "denser" than the real number line, but also it is the case that the former is categorically longer than the latter!

At this point, it is worth our attention to examine one perhaps counter-intuitive feature concerning the topology of $\mathbb{H}$. While so far we know that the hyperreal number line is both denser and longer than the real number line, (meaning that there should be a lot more elements in $\mathbb{H}$ than in $\mathbb{R}$), it turns out that they actually share the

same "size".

The "size" of sets may be measured by the formal mathematical concept of cardinality (or countability), which is in turn represented by some cardinal numbers. While for finite sets the cardinality is simply the finite natural number that is equal to the number of elements in the set (in which case the corresponding cardinal number is identical to the normal finite natural number), the situation becomes bizarre and interesting once we aim at infinite sets. To begin with, it has been proven that the cardinality of $\mathbb{N}$ is the smallest infinite cardinality, and the corresponding cardinal number is denoted as $\aleph_0$, which is also the smallest transfinite cardinal. Once we accept the *continuum hypothesis* (which we will, considering that the *Axiomatic Set theory*, the context where the axiom is derived, is beyond the scope and purpose of this thesis), we can then readily assume that the next cardinality of an infinite set (or the second smallest one) would be that of $\mathbb{R}$, the corresponding transfinite cardinal of which is denoted as $\aleph_1 = 2^{\aleph_0}$. Notice that the *continuum hypothesis* guarantees that there does not exist any cardinal number between $\aleph_0$ and $\aleph_1$. Now, equipped with this basic assumption in cardinality, we are ready to investigate the cardinality of $\mathbb{H}$.[36]

Theorem 3.2.2: The cardinality of $\mathbb{H}$, denoted as $|\mathbb{H}|$, is equal to that of $\mathbb{R}$. In other words, $|\mathbb{H}| = |\mathbb{R}|$.[37]

Proof. The general strategy of the proof is to find some deliberately designed upper and lower bound of $|\mathbb{H}|$. On the one hand, since we know there are definitely more

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[36]Again, more details of the cardinalities of $\mathbb{R}$, $\mathbb{N}$, and the *continuum hypothesis* can be found in sections 2.1 and 2.2 of [6].

[37]The proof technique has been explicitly accounted for by Asaf Karagila on the Mathematics stack exchange forum. Details of Karagila's insight can be found at: http://math.stackexchange.com/questions/54059. Here I especially thank Karagila for his contribution.

elements in $\mathbb{H}$ than in $\mathbb{R}$, we can deduce:

$$(1) \ |\mathbb{R}| \leq |\mathbb{H}|.^{38}$$

On the other hand, since $\mathbb{H}$ is a quotient set of $\mathbb{R}^{\mathbb{N}}$,[39] we may deduce:

$$(2) \ |\mathbb{H}| \leq |\mathbb{R}^{\mathbb{N}}| = \aleph_1^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.^{40}$$

To combine the results from $(1), (2)$, we have:

$$|\mathbb{R}| \leq |\mathbb{H}| \leq |\mathbb{R}^{\mathbb{N}}|.$$

Since we know $|\mathbb{R}| = |\mathbb{R}^{\mathbb{N}}| = 2^{\aleph_0}$, we can deduce that:

$$2^{\aleph_0} \leq |\mathbb{H}| \leq 2^{\aleph_0}.$$

And thus we can conclude:

$$|\mathbb{H}| = |\mathbb{R}| = |\mathbb{R}^{\mathbb{N}}| = 2^{\aleph_0}.$$

$\square$

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[38]Notice that the fact that there are more elements in $\mathbb{H}$ does not guarantee $|\mathbb{H}|$ to be strictly bigger. Indeed, such situation can happen very often when the scale of sets goes to infinity. A simple example to show this feature of infinite cardinalty is: whereas $\{i \in \mathbb{N} : i = 2 \cdot k, k \in \mathbb{N}\} \subset \mathbb{N}$, $|\{i \in \mathbb{N} : i = 2 \cdot k, k \in \mathbb{N}\}| = |\mathbb{N}|$.

[39]This is further explained in the appendix.

[40]It is the case that $\aleph_0 \cdot \aleph_0 = \aleph_0$, because the cardinality of a Cartesian product of two countable sets is still countable. See Theorem 3.7 in [6].

3.3   Abstract Algebraic Properties of Hyperreals

In this section, the central goal is to show that: algebraically, $\mathbb{H}$ is an *ordered field* which contains $\mathbb{R}$ as a *proper subfield*. To proceed, I will assume some basic knowledge of abstract algebra. In particular, I assume the known definition of *ring*.[41] Based on the assumed knowledge of a ring, here I will provide some basic definitions:

Definition 3.3.1 [3]

Given a ring with two binary operations: addition, $+$, and multiplication, $\times$:

- A *field*, denoted as $(F, +, \times)$, is a commutative ring (that is, a ring whose multiplicative operation is commutative) with unity (or multiplicative identity), where each nonzero element has a multiplicative inverse and is considered as a *unit*.

- An *ordered field* is a field with a total ordering (as defined in the last section), denoted as $(F, +, \times, \leq)$ for some $a, b, c \in F$ such that:

  1. if $a \leq b$, then $a + c \leq b + c$;

  2. if $0 \leq a$ and $0 \leq b$, then $0 \leq a \times b$.

To proceed, I will first clarify the algebraic structure of $\mathbb{R}$, and again with the critical assistance of the Transfer Principle, I will then prove the desired algebraic structure of $\mathbb{H}$.

Claim 3.3.1: $\mathbb{R}$ is an ordered field.

Since Claim 3.3.1 is a well-known fact concerning the algebraic structure of $\mathbb{R}$ (in fact, the abstract notion of ordered field is supposedly first derived from the structure of $\mathbb{R}$, which serves as the blueprint of all ordered fields), my proof will be concise.

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[41]Note that a precise definition of ring can be found in Chapter 12, [3].

Proof. First, we need to show $\mathbb{R}$ is a field, with the associated binary operations of standard addition and multiplication. Let a, b, and c be generic elements in $\mathbb{R}$. Claim: the additive identity is 0 and the unity is 1.

Clearly, it is the case that:

- Addition is commutative, since $\forall a, b[a + b = b + a]$.

- Addition is associative, since $\forall a, b, c[(a + b) + c = a + (b + c)]$.

- 0 is the additive identity, since $\forall a[a + 0 = 0 + a = a]$.

- An Additive inverse always exists, since $\forall a, \exists (-a)[a + (-a) = 0]$.

- Multiplication is associative, since $\forall a, b, c[(a \times b) \times c = a \times (b \times c)]$.

- Multiplication is distributive, since $\forall a, b, c[(a + b) \times c = a \times c + b \times c]$.

Based on the definition of ring, the above is sufficient to show $\mathbb{R}$ is a ring [3].
In addition,

- $\mathbb{R}$ is a commutative ring i.e. the multiplication is commutative in $\mathbb{R}$, since $\forall a, b[a \times b = b \times a]$.

- Every nonzero element is a unit, since $\forall a \neq 0, \exists \frac{1}{a}[a \times \frac{1}{a} = 1]$.

Based on the definition of a field, we conclude that $\mathbb{R}$ is a field.

Next, to finish up the proof, we need to show $\mathbb{R}$ is an ordered field. Since from the last section we have already known that $\mathbb{R}$ has a total ordering $\leq$, we need to show it suffices its role in $\mathbb{R}$ insofar as a field. Indeed, it is clearly the case that:

- $(a \leq b) \rightarrow (a + c \leq b + c)$;

- $(0 \leq a \wedge 0 \leq b) \rightarrow (0 \leq a \times b)$.

Therefore, based on the definition of ordered field, we can conclude that $\mathbb{R}$ is an ordered field.

<div align="right">□</div>

At this point, we are ready to deduce the following theorem [4].

Theorem 3.3.1: The structure $S_{\mathbb{H}} = (\mathbb{H}, +, \times, \leq)$ makes $\mathbb{H}$ an *ordered field*, which contains $\mathbb{R}$ as a *proper subfield*.

Proof. Intuitively, I claim that the additive identity in $\mathbb{H}$ is still 0, i.e. $(0, 0, 0, ..., 0, ...)$, and the multiplicative identity in $\mathbb{H}$ is still 1, i.e. $(1, 1, 1, ..., 1, ...)$. While we could potentially go over the similar proof as the proof in the case of $\mathbb{R}$ just provided (among proving many properties, we need to test for example: given $h = (a_1, a_2, a_3, ..., a_i, ...)$ where $a_i \in \mathbb{R}$, the additive inverse is $-h = (-a_1, -a_2, -a_3, ..., -a_i, ...)$ and if $h \neq (0, 0, 0, ..., 0, ...)$ the multiplicative inverse is $h^{-1} = (a_1^{-1}, a_2^{-1}, a_3^{-1}, ..., a_i^{-1}, ...)$),[42] we need not do so. For, considering in the previous of proof of $\mathbb{R}$ where *all* defining properties that constitute an ordered field can be expressed in first order language and $\mathbb{R}$ has been proven to satisfy all of them, according to the Transfer Principle, we may conclude directly that every single one of those properties must also hold in $\mathbb{H}$! Therefore, without going into the tedious procedure again, we can conclude immediately that $\mathbb{H}$ is also an ordered field. To finish up the proof, we still need to show $\mathbb{R}$ is contained in $\mathbb{H}$ as a proper subfield. This should be trivial at this point: Since we know $\mathbb{R} \subset \mathbb{H}$ (or $\mathbb{R}$ is a proper subset of $\mathbb{H}$) and both $\mathbb{R}$ and $\mathbb{H}$ are fields, this amounts to showing that $\mathbb{H}$ contains $\mathbb{R}$ as its *proper subfield*.

<div align="right">□</div>

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[42]Notice that in fact it is not necessary for every a_i in $h \neq 0$ to have a multiplicative inverse a_i^{-1}. For, the condition that $h \neq 0$ only guarantees $\{i \in \mathbb{N} : a_i \neq 0\}$ is Quasi-Big (or only a Quasi-Big set of a_i are nonzero). As a result, there could well be $a_j = 0$, in which case $\nexists a_j^{-1}$. Thus, to avoid ambiguity in the notation h^{-1}, we could let the term "a_j^{-1}" where $a_j = 0$ to be simply always 0 (again, notice that such notification is insignificant, since what matters for a hyperreal number is only a Quasi-Big set of its real terms).

Remarks on Theorem 3.3.1:

One further step we could take here to show the relationship between the algebraic structure of $\mathbb{R}$ and that of $\mathbb{H}$ in addition to the containment as fields is to show that the respective total ordering in each structure is consistent. According to [4], such ordering-preserving feature between $\mathbb{R}$ and $\mathbb{H}$ may be formulated as the following observation.

Observation after Theorem 3.3.1 [4]: Define the map M: $\mathbb{R} \longrightarrow \mathbb{H}$ such that given $r \in \mathbb{R}$, $M(r) = (r, r, r, r, .., r, ...)$. Then, M is considered an *ordering-preserving field isomorphism* if, given $r, s \in \mathbb{R}$,

- $M(r+s) = M(r) + M(s)$. This is satisfied, since $(r+s, r+s, r+s, .., r+s, ...) = (r, r, r, .., r, ...) + (s, s, s, .., s, ...)$;

- $M(r \times s) = M(r) \times M(s)$, since $(r \times s, r \times s, r \times s, .., r \times s, ...) = (r, r, r, .., r, ...) \times (s, s, s, .., s, ...)$;

- $M(r) \leq M(s)$ iff $r \leq s$. This is satisfied, since $(r, r, r, .., r, ...) \leq (s, s, s, .., s, ...)$ iff $r \leq s$.

To end this section, I will provide one last and provisional comment concerning the algebraic structure of $\mathbb{H}$. While in this thesis the construction of $\mathbb{H}$ is primarily based upon the tools and notions from mathematical logic and set theory (which is the most common fashion of construction up to the date), it need not be so. In fact, Benci (et al.) has managed to provide a purely algebraic construction of $\mathbb{H}$. Given a field F, it can be proven that F suffices to play the role of the hyperreal number system *iff* it is homomorphic to (meaning there exists a mapping that preserves both operations of the ring/field) a special category of rings of real-valued functions (the details can be found in [2]).

Conclusion

Admittedly, the construction of hyperreal number system is purely conceptual. In other words, the construction of $\mathbb{H}$ at its core is a matter of some manipulation of logical symbols and setting of rules. As a result, one might not be able to see the actual utility of $\mathbb{H}$. Nevertheless, it is worth emphasising that: while this thesis mostly focuses on providing a complete and rigorous account of the hyperreal number system itself, one major motivation for developing $\mathbb{H}$ at the first place is to do *nonstandard analysis.* As opposed to the traditional method, i.e. real analysis, nonstandard analysis can be seen as an alternative method to do analysis which is equipped with powerful new concepts derived from infinitesimals and infinities. Indeed, since its original creation by Robinson, nonstandard analysis has been adopted and applied to solve practical problems most noticeably in the areas of engineering and economics. Again, considering the aim and scope of this thesis, I am mainly concerned with clarifying the basic conceptual frame, i.e. hyperreal number system, through which nonstandard analysis becomes viable. In other words, as it has been shown through this thesis, hyperreal number system is after all not a vacuous language game, but, on the contrary, a coherent and inclusive conceptual frame.

To end the conclusion, I will once again address the question concerning the value and meaning of hyperreal number system—namely, *Why would this entirely new conceptual frame be something worth engaging while we could simply be contented with the already well-established tradition and techniques of standard real analysis?*

To reply, there are two aspects in response to this concern: first, while real analysis and nonstandard analysis are indeed intrinsically both conceptual frameworks that need not have any corresponding factual ground for their existences (which makes the choice seem to be arbitrary), nonstandard analysis speaks more directly to our ordinary intuition and hence may be grasped relatively easier; second, as it has been shown, it is the inclusiveness of the hyperreal number system, both in terms of the numbers it contains and in terms of the properties it preserves, that makes $\mathbb{H}$ a highly workable system of numbers and the nonstandard analysis an extraordinary way to do mathematical analysis. Indeed, to some extent, it is perhaps the case as what *Gödel* predicts: *"There are good reasons to believe that nonstandard analysis, in some version or other, will be the analysis of the future."* [4]

Appendix: Filter, Quasi-Big Set, and Ultrapower Construction of $\mathbb{H}$

In chapter 2, I present a construction of $\mathbb{H}$, where each hyperreal can be represented as a sequence of reals indexed by $\mathbb{N}$. In particular to further conceptualise, given a hyperreal number $h = (r_1, r_2, r_3, ..., r_i, ...)$ where $i \in \mathbb{N}$ and $r_i \in \mathbb{R}$, observe that

$$h \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times ...$$

or

$$h \in \mathbb{R}^{\mathbb{N}}$$

However, it is worth mentioning that $\mathbb{H}$ is not the same as $\mathbb{R}^{\mathbb{N}}$ as a result of the equivalence relation induced in $\mathbb{H}$. To clarify, according to the equivalence relation (or $\equiv$-equivalence) defined in $\mathbb{H}$, two hyperreals differ only when a *Quasi-Big* set of real components differ. For instance, whereas, in $\mathbb{R}^{\mathbb{N}}$, $h_1 = (1, 2, 2, 2, ...2, ...)$ and $h_2 = (2, 2, 2, ..., 2, ...)$ are two different elements due to the different value in the first location of the two sequences, in $\mathbb{H}$ however $h_1 = h_2$, since they have a cofinite, and thus Quasi-Big set, of elements in common despite the trivial difference at the first location. In fact, because of the inducement of the equivalence relation in $\mathbb{H}$, each distinctive hyperreal is seen as a distinctive equivalent class in $\mathbb{R}^{\mathbb{N}}$ (which forms a

partition of $\mathbb{R}^{\mathbb{N}}$), and therefore $\mathbb{H}$ is considered as the set of all equivalence classes in $\mathbb{R}^{\mathbb{N}}$ based on $\equiv$-equivalence.[43] In other words, $\mathbb{H}$ is the quotient set of $\mathbb{R}^{\mathbb{N}}$ induced by $\equiv$-equivalence. This can be denoted as:

$$\mathbb{H} = [\mathbb{R}^{\mathbb{N}}/\equiv]$$

In the rest of the appendix, I will present the formal conceptual foundation behind the key of the equivalence relation, i.e. the notion of Quasi-Big set, in $\mathbb{H}$. Hopefully, such efforts will help to further demonstrate the nature of the hyperreal number system.

The idea of Quasi-Big sets that is employed to induce the equivalence relation in $\mathbb{H}$ can be traced back to the concept of *filter*. To start the investigation, some basic definitions are presented as follow.

Definition 1. Filter, Proper Filter, and Ultrafilter[4], [7]:

Let I be a nonempty set of elements.

1.1 A *filter* over I, denoted as F, is a set of subsets of I such that:[44]

- F is closed under supersets: if $A \in F$ and $A \subseteq B \subseteq I$, then $B \in F$.

- F is closed under finite intersections: if $A, B \in F$, then $A \cap B \in F$.

1.2 A *proper filter* over I is a *filter* F_p that does not contain the empty set, i.e. $\varnothing \in F_p$.

1.3 An *ultrafilter* is a *proper filter* U such that:

$$\forall A \subseteq I, (A \in U) \veebar (A^c \in U).$$

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[43]Observe that the idea to view distinct numbers as distinct equivalence classes is not new. In fact, recall the Cauchy sequence construction of $\mathbb{R}$, according to which a real number can be defined as an equivalence class of Cauchy sequences of rational numbers based on Cauchy equivalence. A simple explanation of this can be found in 1.3 of [4].

[44]In other words, F is a subset of the power set of I.

Remarks Concerning Definition 1

Notice that what eventually matters for our purpose to understand *Quasi-Big* sets is a special category of *ultrafilters*. To proceed, we need to first prove the following claim as a result of Definition 1.

Claim: Given $i \in I$, a *principal ultrafilter generated by i*, namely: $U^i = \{A \subseteq I : i \in A\}$, is indeed an *ultrafilter* on I.

Proof. To begin with, notice that U^i is essentially the set of all subsets in I that contain a fixed element $i \in I$.

Therefore, we know U^i is not empty, since at least the single element set $\{i\} \in U^i$.

In addition, given $A, B \in U^i$, we know $i \in A$ and $i \in B$, and thus $i \in A \cap B$. Therefore, $A \cap B \in U^i$ (U^i is closed under finite intersections).

What is more, given $A \in U^i$, and $A \subseteq B \subseteq I$, we know $i \in A$; and since $A \subseteq B \subseteq I$, we know $i \in B$. Therefore, $B \in U^i$ (U^i is closed under supersets).

From above, we can conclude that U^i is a *proper filter*.

Last, it is obvious that $\forall A \subseteq I, (i \in A) \veebar (i \in A^c)$. Therefore, $\forall A \subseteq I, (A \in U^i) \veebar (A^c \in U^i)$.

Thus, we can conclude U^i is an *ultrafilter*. $\qquad\square$

Based on the claim concerning *principal ultrafilter generated by i*, a *nonprincipal ultrafilter* can be defined as any *ultrafilter* that is not *principal*. Notice that it is upon this notion of *nonprincipal ultrafilter* that *Quasi-Big* sets are directly founded. However, before proceeding to show the connection between *nonprincipal ultrafilter* and *Quasi-Big* sets, one simple but important claim needs to be made. Namely, given an *ultrafilter* U, it can be proven that if U contains a finite set, then it must be *principal*. As a result, a *nonprincipal ultrafilter* insofar as the negation of *principal ultrafilter* must contain only infinite sets.

Observation: The set of all *Quasi-Big* sets U^Q is equivalent to the definition of *nonprincipal ultrafilter* on $\mathbb{N}$. In other words, given any fixed *nonprincipal ultrafilter* U on $\mathbb{N}$, U can induce an eligible equivalence relation in $\mathbb{R}^{\mathbb{N}}$ to construct $\mathbb{H}$. Thus, $\mathbb{H}$ through this construction is also known as the *ultrapower* of $\mathbb{R}$, namely the quotient set of $\mathbb{R}^{\mathbb{N}}$ induced by the equivalence relation defined by an *ultrafilter* [4].[45]

Proof. The proof should be straightforward at this point, simply by invoking the definition of *Quasi-Big* sets.

- clearly, U^Q consists of some subsets of $\mathbb{N}$.

- From Definition 2.3.2, we know if $A, B \in U^Q$, then $A \cap B \in U^Q$. Hence U^Q is closed under intersection.

- From Definition 2.3.3, we know if $A \in U^Q$, and $A \subseteq B \subseteq \mathbb{N}$, then $B \in U^Q$. Hence Hence U^Q is closed under superset.

- From Definition 2.3.4, we know $\forall A \subseteq \mathbb{N}, (A \in U^Q) \veebar (A^c \in U^Q)$.

Hence, U^Q is an *ultrafilter*. Furthermore, from Definition 2.3.1, we know *Quasi-Big* sets must be infinite. Therefore, we can conclude that *Quasi-Big* sets U^Q indeed forms a *nonprincipal ultrafilter* on $\mathbb{N}$.

$\square$

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[45]It is worth mentioning that *Zorn's Lemma* guarantees the existence of *nonprincipal ultrafilters* on any *infinite set*. As a result, this would assure us that U^Q must exist and the *ultrapower* construction of $\mathbb{H}$ must be possible, since $\mathbb{N}$ is clearly an infinite set. Details of *Zorn's Lemma* can be found in [4].

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