Perfect Colorings of a Design With 2-Dimensional Euclidean Crystallographic Symmetry Group

BY

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A Study

Presented to the Faculty

of

Wheaton College

in Partial Fulfillment of the Requirements

for

Graduation with Departmental Honors

in Mathematics

Norton, Massachusetts

May 2018

Acknowledgments

First of all, I would like to offer my sincerest thanks to my amazing thesis advisor Professor Rachelle DeCoste. Rachelle's faith in me and her constant encouragements are the biggest motivations for this work. I cannot emphasize how grateful I am for her help and patience.

I would like to thank Professor Shelly Leibowitz and Professor Chris Kotyk for being my committee members. They both have been very supportive and positive to me and my work.

I would also like to thank my parents and friends for making my days at Wheaton fulfilling and exciting.

Finally, I would like to offer my special thanks to my boyfriend Weiqi Feng for his love and support. I want to wish him the best of luck for his thesis next year. Hope he will have as much fun as I had!

Contents

1 Abstract

This thesis introduces perfect colorings and the systematic way of generating perfect colorings. We study the perfect colorings for designs whose symmetry groups are 2-dimensional Euclidean crystallographic groups. Two-dimensional Euclidean crystallographic groups are discrete subgroups of the isometry group of the Euclidean plane. We classify crystallographic groups by their ranks. Crystallographic groups of rank 0, rank 1, and rank 2 are explained with details and examples. Theorems that related to the number of perfect colorings for designs with crystallographic symmetry groups are constructed and proved.

2 Background

In this paper, we study colorings of objects with symmetry. We begin with objects with only rotation and reflection symmetries, and then expand our study to include translations in one direction and then two directions. In order to study symmetry with rigor, we begin with some background ideas.

In this chapter, we introduce background and give several definitions that build the foundations for the reminder of this study. Examples are provided to explain the definitions as necessary.

2.1 Symmetry and Symmetry Group

Definition 2.1 *Symmetries* are transformations, including reflection, rotation or translation, under which an object is invariant to itself. It is well known that the set of such transformations for an object forms a group. Thus, the *symmetry group* of an object is the group of all symmetries of the object.

Example 2.2 A square has 8 symmetries: rotations around the center by, 0, $\pi/2$, π, and $3\pi/2$ radians counter-clockwise, and reflections over the horizontal axis, the vertical axis, and the two diagonal axes. The set containing the 8 symmetries forms the symmetry group of the square. This group is *the dihedral group* D_8 *of order 8*, a well understood group. Since it is a group, the composition of any two of these symmetries is again one of the 8 symmetries. Also, for every symmetry g, there is an inverse symmetry g^{-1}. This means that the composition of any symmetry and its inverse symmetry is the identity symmetry, rotation by 0 radians. Clearly, for the reflections, each is its own inverse symmetry. Also the inverse symmetry of rotation by $\pi/2$ is rotation by $3\pi/2$.

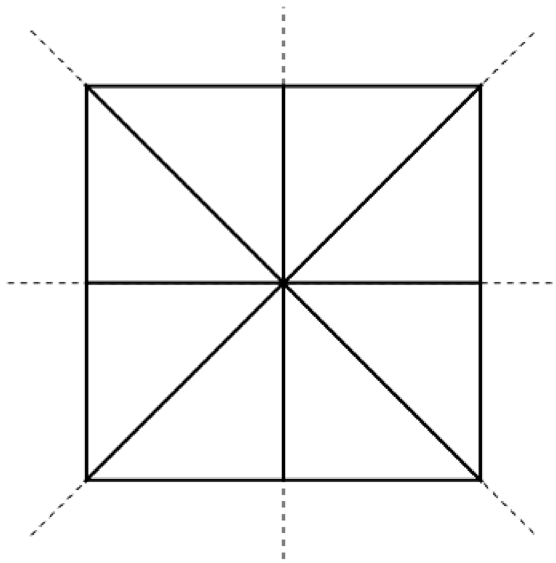


Figure 1: A square with its reflection axes.

2.2 Metric Space and Isometry Group

Definition 2.3 [1] A *metric* is a function D that defines a distance between each pair of elements of a set. Given any set $\mathbb{P}$, a function $D : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ is a metric if the following properties are satisfied for every $P, Q \in \mathbb{P}$:

(i) $D(P, Q) = D(Q, P)$;

(ii) $D(P, Q) \geq 0$;

(iii) $D(P, Q) = 0$ if and only if $P = Q$.

Example 2.4 In the Euclidean plane, denoted $\mathbb{E}^2$, the distance between 2 points (x_1, y_1) and (x_2, y_2) is defined by $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The distance function d is a metric because d satisfies the three properties in the Definition 2.3.

(i) For every (x_1, y_1) and (x_2, y_2), $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d((x_2, y_2), (x_1, y_1))$;

(ii) For every (x_1, y_1) and (x_2, y_2), $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \geq 0$;

(iii) $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0$ if and only if $x_2 - x_1 = 0$ and $y_2 - y_1 = 0$ if and only if $x_1 = x_2$ and $y_1 = y_2$ if and only if $(x_1, y_1) = (x_2, y_2)$.

This metric d on $\mathbb{E}^2$ is called the *Euclidean metric.*

Definition 2.5 A *metric space* M is a set in which distances between all pairs of elements are defined by a metric.

The Euclidean plane can be considered as a metric space with the Euclidean metric d as its metric.

Definition 2.6 Given a metric space M with its metric D, an *isometry* is a one-to-one and onto mapping $T : M \to M$ that preserves distance, i.e., $D(x, y) = D(T(x), T(y)) \; \forall \; x, y \in M$.

It is well known that the set of all isometries of a metric space M forms a group under composition.[2] The *isometry group* of M is denoted by $\text{Isom}(M)$. It is proved in [1] and many other sources that the composition of two isometries is an isometry and the inverse of an isometry is an isometry. The identity mapping of $\text{Isom}(M)$ is $T_e(x) = x \; \forall \; x \in M$. The isometry group of the Euclidean plane is denoted by $\text{Isom}(\mathbb{E}^2)$, and the identity mapping of $\text{Isom}(\mathbb{E}^2)$ is $T_e(x, y) = (x, y) \; \forall \; (x, y) \in \mathbb{E}^2$.

Let $\vec{v}$ be a vector in the Euclidean plane and let (x, y) be a generic point in $\mathbb{E}^2$, then the mapping $T(x, y) = (x, y) + \vec{v}$ is a isometry that acts as a translation. In math, translation is an operation that shifts an object by a fixed distance. For the mapping $T(x, y) = (x, y) + \vec{v}$, the point (x, y) is translated by distance $||\vec{v}||$ in the direction of $\vec{v}$.

It is shown in [2] that reflections across any line $l \in \mathbb{E}^2$ and rotations about any point $(x, y) \in \mathbb{E}^2$ are isometries of the Euclidean plane. The next two examples illustrate that reflections and rotations are elements of $\mathrm{Isom}(\mathbb{E}^2)$.

Example 2.7 Let l be any line in the Euclidean plane, and P, Q be any two points in the Euclidean plane. Reflecting P and Q across the mirror line l gives points P' and Q', respectively. The reflection across the line l is an isometry of the Euclidean plane. The distance between P and Q and the distance between the reflection points P' and Q' are equal.

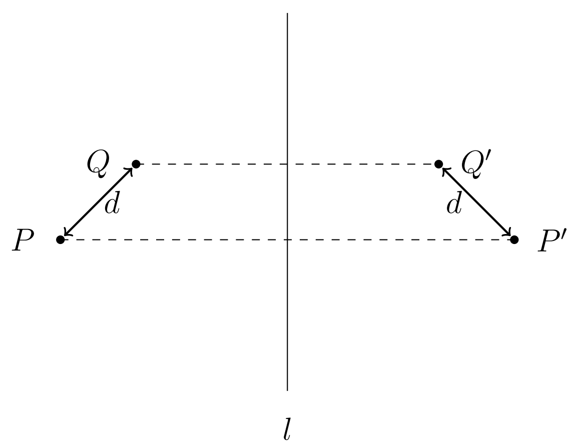


Figure 2: Points P' and Q' are reflections of points P and Q, respectively, across the mirror line l.

Example 2.8 Let O, P, and Q be any points in the Euclidean plane. Rotate P and Q by a fixed angle with O as the rotation center. Let the end results be points P' and Q', respectively. Rotation by a fixed angle with a fixed rotation center is an isometry of the Euclidean plane. The distance between P and Q and the distance between the rotation points P' and Q' are equal.

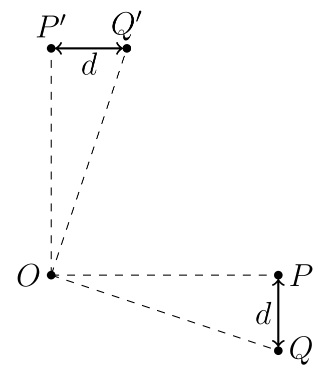


Figure 3: Points P and Q are rotated by a fixed angle with O as the rotation center. The end results are points P' and Q', respectively.

2.3 Crystallographic Groups

Definition 2.9 The *orthogonal group* $O(2)$ is defined as $O(2) = \{M : M \text{ is a } 2 \times 2 \text{ matrix with real entries such that } MM^T = I\}$, where I is the identity matrix.

Given any isometry $A : \mathbb{E}^2 \to \mathbb{E}^2$, it is proved in [2] that there exists a vector $\vec{w} \in \mathbb{R}^2$ and an orthogonal matrix B such that $A(x) = Bx + \vec{w}\ \forall\ x \in \mathbb{R}^2$. Let ϕ be a mapping from $\text{Isom}(\mathbb{E}^2)$ to $O(2)$ defined by $\phi(A) = B$. Then it is straightforward to show that ϕ is a group homomorphism, and its kernel is the group of all translations of $\mathbb{E}^2$, $\text{Trans}(\mathbb{E}^2)$.

A 2-dimensional Euclidean *crystallographic group* G is a discrete subgroup of $\text{Isom}(\mathbb{E}^2)$. Then the group homomorphism $\phi : G \to O(2)$, defined as above but restricted to G, has $G \cap \text{Trans}(\mathbb{E}^2)$ as its kernel. The kernel $G \cap \text{Trans}(\mathbb{E}^2)$ forms a lattice $\Lambda \in \text{Isom}(\mathbb{E}^2)$, which is a discrete subgroup of translations. The image of the restricted group homomorphism, $\phi(G)$, forms the *point group* $\overline{G}$ of the crystallographic group G. Then by group theory, $G \cong \overline{G} \times G \cap \text{Trans}(\mathbb{E}^2)$. Thus the crystallographic group G is constructed by combining a lattice of translations with the point group. [2]

Definition 2.10 The *rank* of a lattice Λ is smallest cardinality of a subset of Λ that generates Λ. The rank of the crystallographic group G is defined as the rank of its lattice Λ.

In the immediately following chapters, we will focus on G of rank 0 and introduce colorings of objects with symmetry groups G of rank 0. When the rank of G is 0, Λ is empty, which means that the crystallographic group does not have translations. Then $G = \overline{G}$ and is finite. Since the group only involves the identity isometry, rotations and reflections, G is either a cyclic group or a dihedral group. Since cyclic groups are subgroups of dihedral groups, we will only focus on dihedral groups in the next few chapters. In Chapter 6 and Chapter 7, we will build on these results and will discuss crystallographic groups of rank 1 and 2, respectively, in the context of colorings.

3 Coloring and Perfect Coloring

In this chapter, we consider objects with symmetry and define a coloring scheme that uses this symmetry. We consider questions of existence and uniqueness of colorings with fixed numbers of colors.

Starting with an uncolored symmetrical pattern, called *design*, we want to color regions of the design corresponding to a symmetry group G of the design, using a finite number of colors. The end result will be referred to as a *coloring*. The entire design may or may not be colored completely.

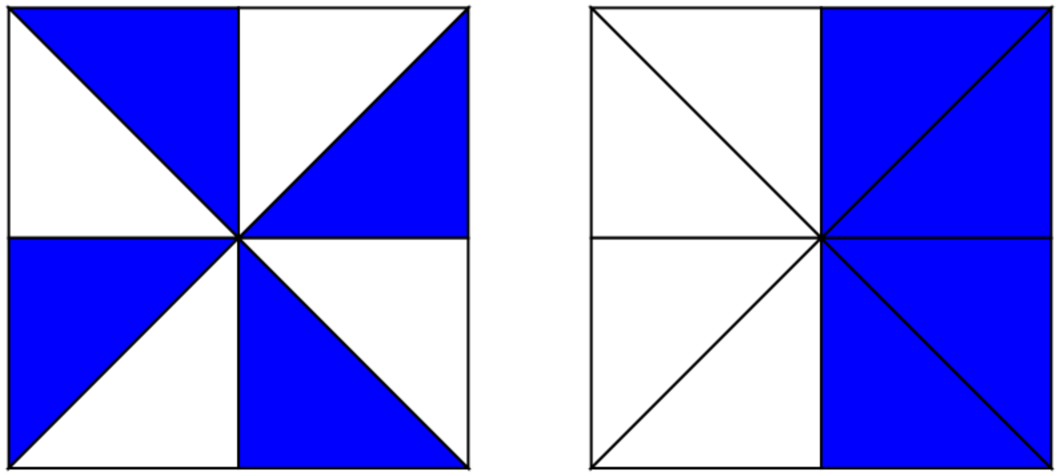
For the following definitions, assume a given design is partitioned into non-overlapping portions which are each assigned a color from a finite set of colors. Let G be the symmetry group for the partitioned design.

Definition 3.1 An element $g \in G$ is called a *color symmetry* if g maps all portions of the colored design colored by one color onto all portions colored by another color. Therefore, g can permute a set of colors.

Definition 3.2 If each $g \in G$ is a color symmetry, then the assignment of colors is called a *symmetric coloring* or *perfect coloring* of the design.

Example 3.3 Consider the colored square in Figure 4(a). This is a perfect coloring because every element $g \in D_8$ is a color symmetry. For example, reflection across either diagonal axis swaps the blue and white, while rotation by $\pi/2$ sends each color to itself. Any of the 8 symmetries in D_8 will result in one of the two situations, i.e., either interchange the two colors or preserve both colors.

Example 3.4 The colored square in Figure 4(b) is not a perfect coloring. You do have $g \in D_8$ such that g is a color symmetry; for example, g = rotation by π



(a) is a perfect coloring. (b) is not a perfect coloring.

Figure 4: Two examples of a colored square.

interchanges all blue and white and $g =$ reflection over the horizontal line sends white to white and blue to blue. However, not all g are color symmetries; for example, under rotation by $\pi/2$ some blue regions are mapped to blue while others are mapped to white.

It is important to note that in a perfect coloring, each color is used to color the same number of fundamental regions. It is not possible for any symmetry element of the symmetry group of the design G to map all portions of the colored design colored by one color onto all portions colored by another color if there exists a portion of the design colored by one color with a different size from another portion of the design.

Let H be a subgroup of G with a finite index. The subgroup H and its cosets provide a way to generate perfect colorings of a design with symmetry group G. In order to do this, we start by finding a collection of fundamental regions in the design such that the regions form a one-to-one correspondence with elements of the symmetry group G. A *sequence of fundamental regions* $\{A_i\}$ for a design with symmetry group G is a set of regions in the design having two properties: (i) The regions are disjoint;

(ii) Given any two regions A_i, A_j, there exists a unique symmetry $g \in G$ such that g maps the region A_i onto the region A_j.

Figure 1 shows a way to divide a square into 8 fundamental regions. Below we will use this as an example of coordinatizing the design using the dihedral group D_8. The way of choosing fundamental regions is not unique. Also, the union of the fundamental regions may or may not cover the entire design. When a design is not completely covered by the fundamental regions, the uncovered portions can be left uncolored. An example of the square with alternate fundamental regions and an uncolored region is shown in Figure 5.

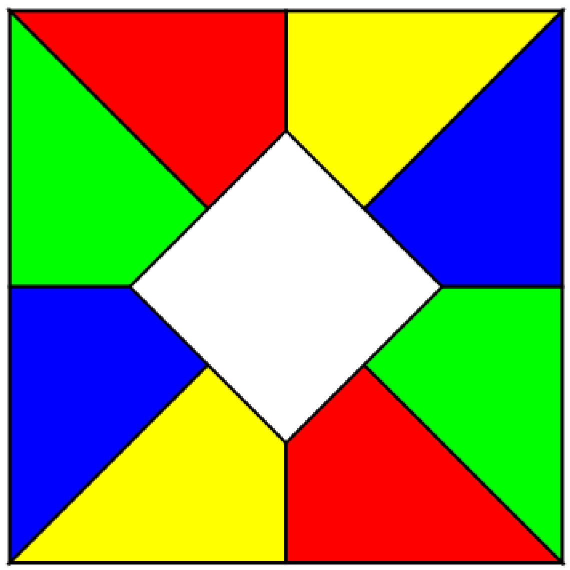


Figure 5: A square with uncolored region.

For D_8, let a be rotation counter-clockwise by $\pi/2$, and b be reflection over the horizontal axis. Then ba^2 is reflection over the vertical axis, and ba and ba^3 are reflections over diagonal axes. Also, a^2 is rotation by π, and a^3 is rotation by $3\pi/2$. Therefore D_8 is generated by a and b and we write $D_8 = \langle a, b \mid a^4 = b^2 = e \rangle = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$.

To coordinatize a design with symmetry group D_{2n}, we can choose any fundamental region to be the beginning region and denote it by $e \in D_{2n}$. Then label the other fundamental regions following the *Basic Principle of Coordinatization* [3], which is that the symmetry h maps the region labelled g onto the region labelled gh. Figure 6 shows a *coordinatized* square.

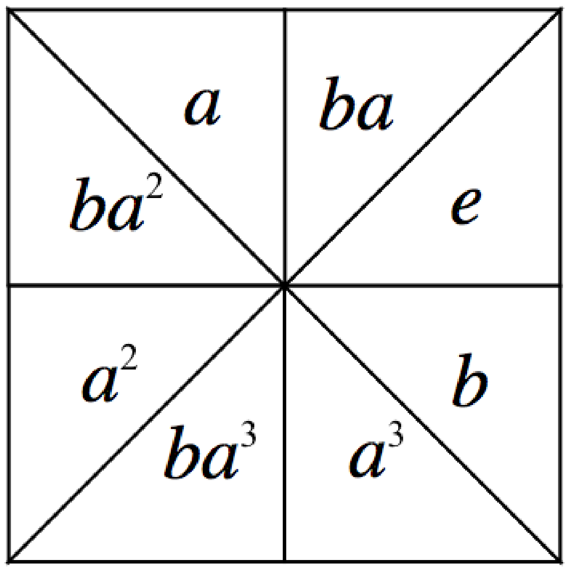
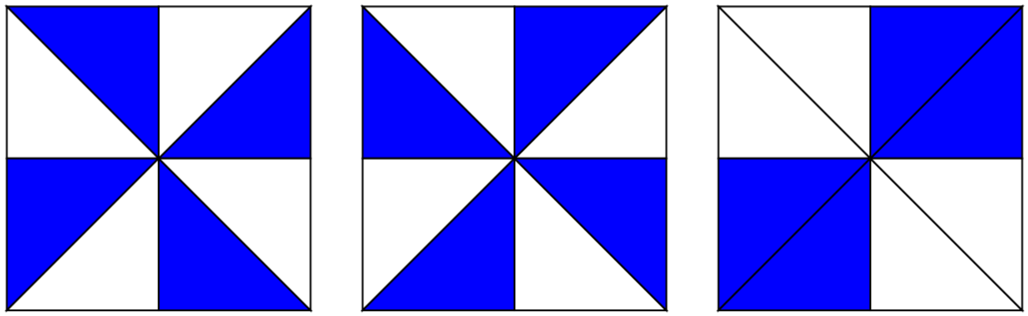


Figure 6: A coordinatized square with elements of D_8.

Let H be a subgroup of D_8, and let Hg represents the right coset of H containing $g \in D_8$. Then the subgroup H and each of its distinct right cosets correspond to a set of fundamental regions of the square. Now we can assign a color to H and a different color to each of the right cosets of H. If H has index n, then we have an n-color perfect colored square.[3] We call a coloring with n colors an *n-coloring.* Figure 9 and Figure 10 are examples of the square colored using this method.

Definition 3.5 Two perfect colorings C_1 and C_2 of a given design are *equivalent* if the design colored by C_1 can be transformed to the design colored by C_2 by (i) applying a symmetry element g of the symmetry group to the design colored by C_1, or (ii) relabeling the colors in C_1, or (iii) combination of (i) and (ii).



(a) (b) (c)

Figure 7: Three perfect colorings of a square.

Example 3.6 The first two perfect colorings shown in Figure 7 are equivalent. We can obtain Figure 7(a) from Figure 7(b) by one of two ways. The first way is applying any of the reflection symmetries. The second option is coloring all white regions with blue and coloring all blue regions with white, i.e., interchange the two colors.

Example 3.7 The perfect coloring shown in Figure 7(c) is inequivalent to the colorings in Figure 7(a) and Figure 7(b). Applying any of the 8 symmetries in the dihedral group of order 8, the symmetry group of a square, to Figure 7(c), we cannot obtain Figure 7(a) or Figure 7(b). Also, interchanging the two colors in Figure 7(c) will not yield Figure 7(a) or Figure 7(b).

4 Perfect Colorings of Square, Hexagon, and Pentagon

In this chapter, we consider examples of perfect colorings of squares, regular hexagons, and regular pentagons. By constructing perfect colorings for regular n-gons with small n, we can study the relationship between perfect colorings of regular n-gon and subgroups of the dihedral group of order $2n$, the symmetry group of a regular n-gon. This relationship will be generalized for all $n \geq 3$, $n \in \mathbb{Z}$, and summarized in the next chapter.

4.1 Perfect Colorings of Square

As stated in Chapter 2, the symmetry group of a square is the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = e \rangle = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$. To construct a perfect coloring of a square, we choose a subgroup H of D_8. For example, let $H = \{e, a^2\}$. The right cosets of H are $Ha = \{a, a^3\}$, $Hb = \{b, ba^2\}$, and $Hba = \{ba, ba^3\}$. Clearly, the subgroup and its three cosets each corresponds to a distinct set of fundamental regions of the coordinatized square. Without loss of generality, we color all fundamental regions that corresponding to the subgroup H with blue, and color fundamental regions that corresponding to Ha with red, Hb with green, and Hba with yellow.[3] The end result is shown in Figure 8.

When we rotate this coloring by 0 or π, all colors are preserved, meaning all regions are still colored with the original color. When the coloring is rotated by $\pi/2$ or $3\pi/2$ radians counter-clockwise, all regions colored with blue are mapped onto regions colored with red, and all regions colored with red are mapped onto regions colored with blue. Similarly, all regions colored with green are mapped onto regions colored with yellow, and all regions colored with yellow are mapped onto regions

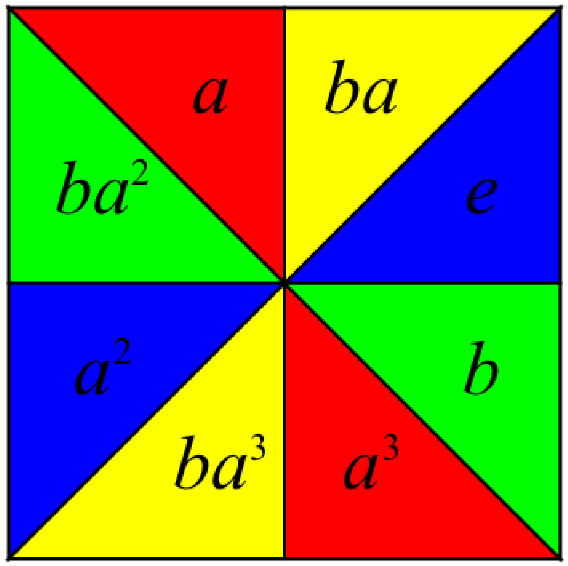
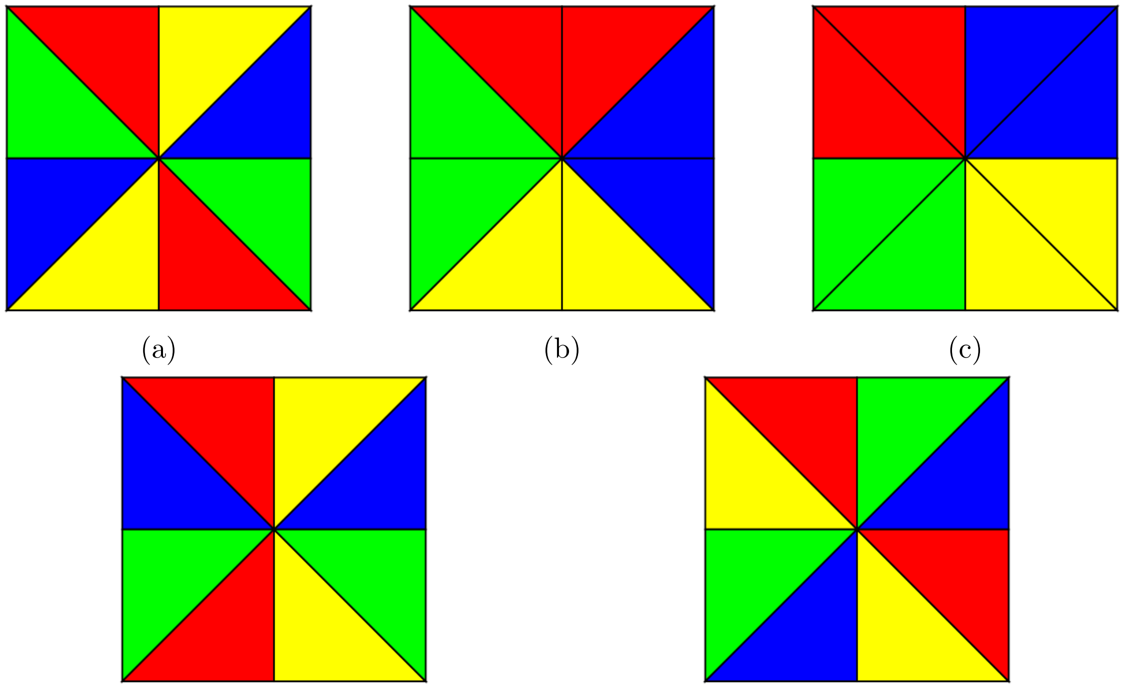


Figure 8: A perfect coloring of the square.

colored with green. When we reflect the coloring across its horizontal or vertical axis, blue regions and green regions are interchanged, and red regions and yellow regions are interchanged. Then when the coloring is reflected across either of its diagonal axes, blue and yellow regions are interchanged, and red and green regions are interchanged. We have checked the 8 symmetries in D_8 and they are all color symmetries. Therefore, we can conclude that this coloring is a perfect coloring.

Coloring the square with 1 or 8 colors is trivial. If the whole square is colored with the same color, any of the 8 symmetry elements in D_8 will preserve the color. If the square is colored with 8 colors, then each of the 8 fundamental regions is colored with a distinct color. Naturally, all symmetries in D_8 are color symmetries. For each case, there exists only 1 possible equivalent coloring.

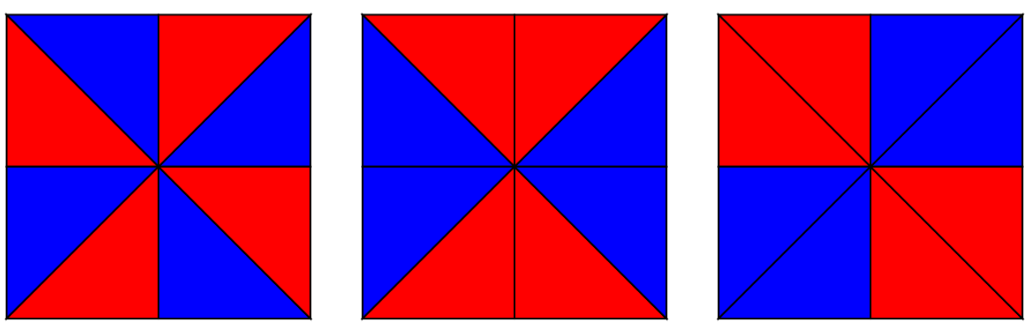
For 4-colorings of a square, we consider subgroups H of D_8 of order 2 as above. D_8 has 5 distinct subgroups of order 2 and each yields an inequivalent perfect coloring. The 5 colored squares are shown in Figure 9.



(a) (b) (c) (d) (e)

Figure 9: The five 4-colored squares. The subgroups used to yield each follows: (a) $H = \{e, a^2\}$; (b) $H = \{e, b\}$; (c) $H = \{e, ba\}$; (d) $H = \{e, ba^2\}$; and (e) $H = \{e, ba^3\}$.

There are 3 inequivalent ways to perfect color the square with 2 colors, obtained using the 3 subgroups of D_8 of order 4. These 3 colored squares are shown in Figure 10 .



(a) (b) (c)

Figure 10: The three 2-colored squares. The subgroups used to yield each follows: (a) $H = \{e, a, a^2, a^3\}$; (b) $H = \{e, a^2, b, ba^2\}$; and (c) $H = \{e, a^2, ba, ba^3\}$.

4.2 Perfect Colorings of Hexagon

Now we use the same scheme to color a regular hexagon. Similar to a square with symmetry group D_8, a hexagon has 12 symmetries including rotations around the center by, 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$ radians counter-clockwise, and reflections over 6 axes. The set containing the 12 symmetries forms the symmetry group of the hexagon. This group is *the dihedral group* D_{12} *of order 12.*

For D_{12}, let a be rotation counter-clockwise by $\pi/3$, and b be reflection over the horizontal axis. Then ba^3 is reflection over the vertical axis, and ba, ba^2, ba^4, and ba^5 are reflections over diagonal axes. Also, a^2 is rotation by $2\pi/3$, a^3 is rotation by π, a^4 is rotation by $4\pi/3$, a^5 is rotation by $5\pi/3$. Therefore $D_{12} = \langle a, b \mid a^6 = b^2 = e \rangle = \{e, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5\}$. Figure 11 shows a coordinatized hexagon with elements of D_{12}.

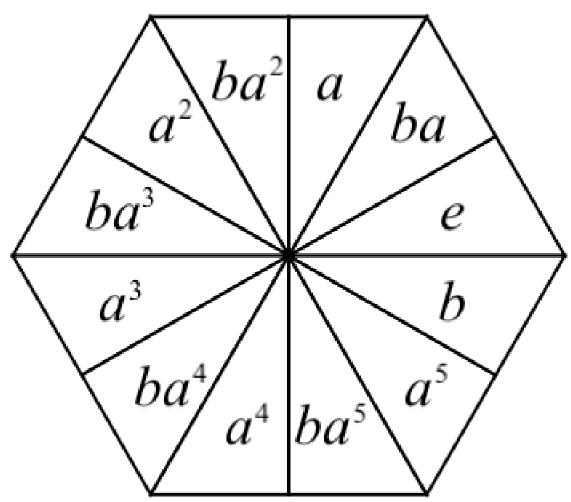


Figure 11: A coordinatized hexagon with elements of D_{12}.

Similar to coloring a square, coloring a hexagon with 1 or 12 colors is trivial. Each case only has 1 possible equivalent coloring. For 6-colorings, we consider subgroups H of D_{12} of order 2 as above. D_{12} has 7 distinct subgroups of order 2 and each yields an inequivalent perfect coloring. The 7 colored hexagons are shown in Figure 12.

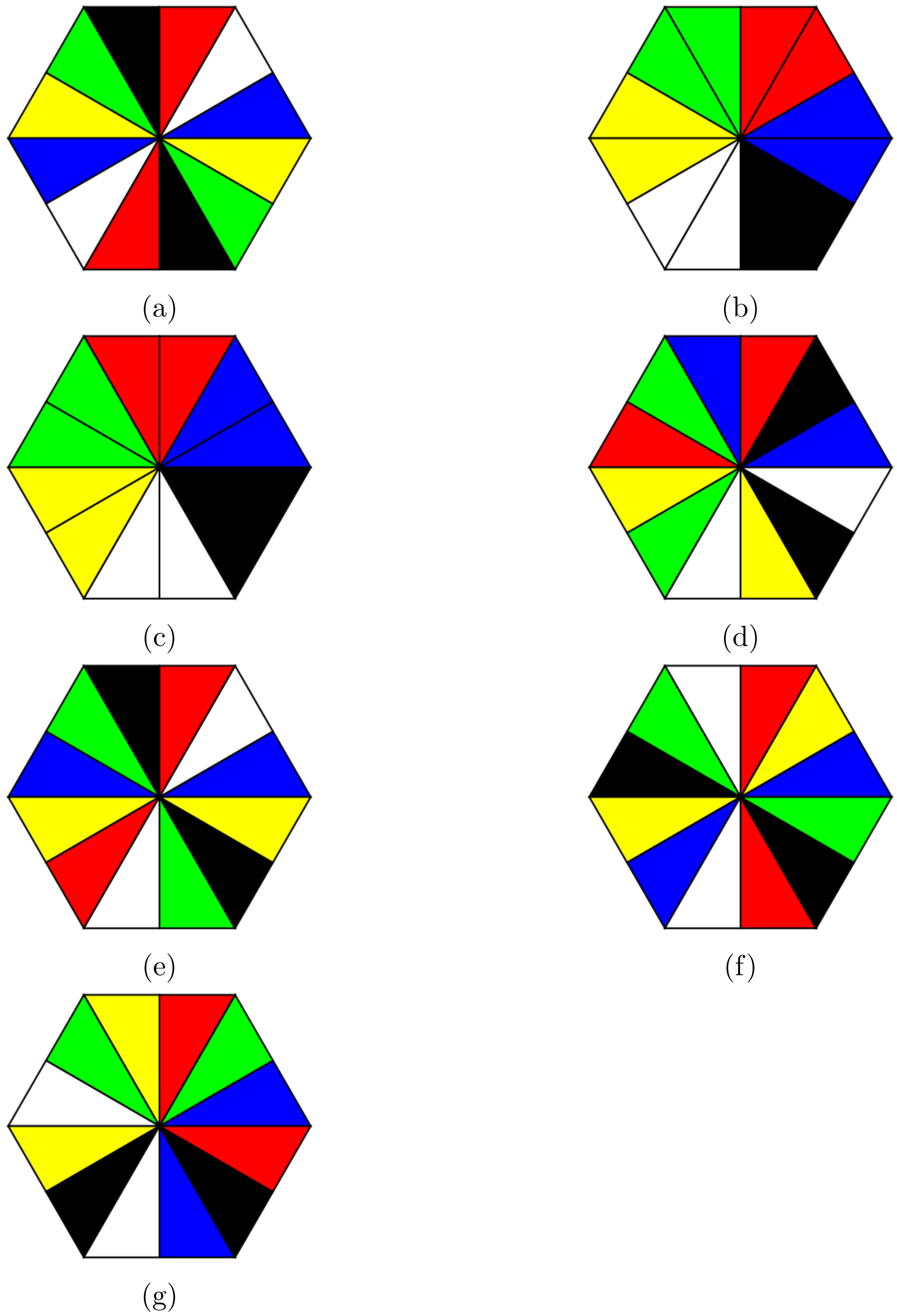


Figure 12: The seven 6-colored hexagons. The subgroups used to yield each follows: (a) $H = \{e, a^3\}$; (b) $H = \{e, b\}$; (c) $H = \{e, ba\}$; (d) $H = \{e, ba^2\}$; (e) $H = \{e, ba^3\}$; (f) $H = \{e, ba^4\}$; and (g) $H = \{e, ba^5\}$.

There is only 1 inequivalent way to perfect color the hexagon with 4 colors, obtained using the unique subgroup of D_{12} of order 3. This 4-coloring hexagon is shown in Figure 13.

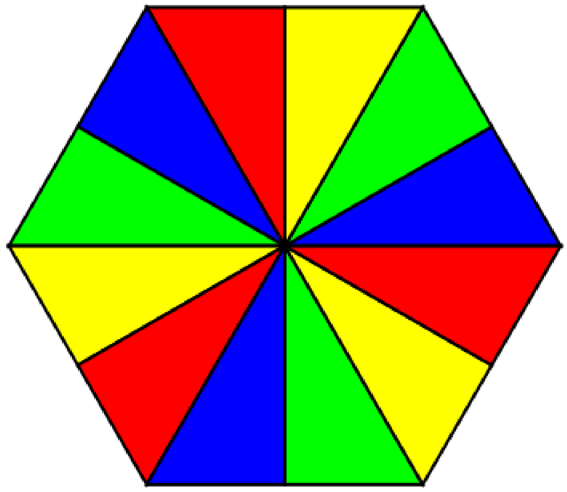
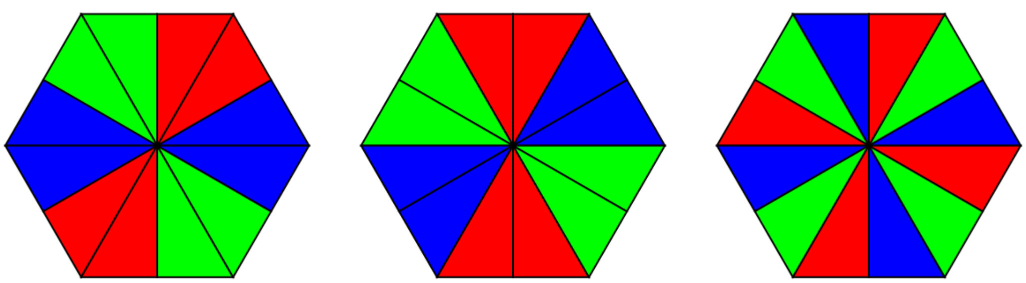


Figure 13: The only 4-colored hexagon. The subgroup used to yield it is $H = \{e, a^2, a^4\}$.

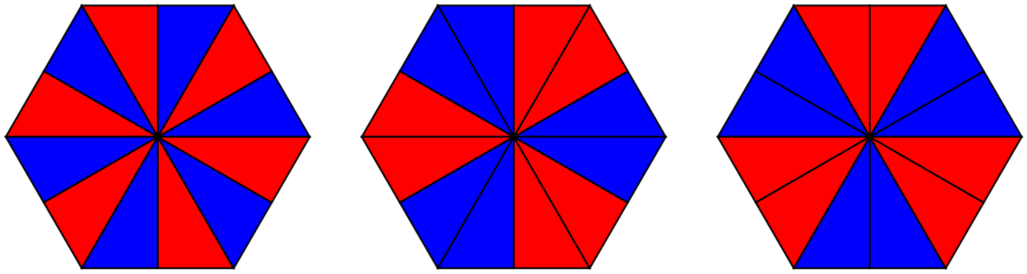
There are 3 inequivalent ways to perfect color the hexagon with 3 colors, obtained using the 3 subgroups of D_{12} of order 4. These 3 colored hexagons are shown in Figure 14.



(a) (b) (c)

Figure 14: The three 3-colored hexagons. The subgroups used to yield each follows: (a) $H = \{e, a^3, b, ba^3\}$; (b) $H = \{e, a^3, ba, ba^4\}$; and (c) $H = \{e, a^3, ba^2, ba^5\}$.

Finally, there are 3 inequivalent ways to perfect color the hexagon with 2 colors, obtained using the 3 subgroups of D_{12} of order 6. These 3 colored hexagons are shown in Figure 15.



(a) (b) (c)

Figure 15: The three 2-colored hexagons. The subgroups used to yield each follows: (a) $H = \{e, a, a^2, a^3, a^4, a^5\}$; (b) $H = \{e, a^2, a^4, b, ba^2, ba^4\}$; and (c) $H = \{e, a^2, a^4, ba, ba^3, ba^5\}$.

As mentioned in the previous chapter, there are colorings that are not perfect. An example of a not perfect coloring of the hexagon is shown in Figure 16. When we rotate this coloring by $\pi/3$ radians counter-clockwise, the fundamental region that corresponding to e will be mapped to the fundamental region a which is colored with the same color. But the fundamental region ba will be mapped to fundamental region ba^2 which is colored with a different color. Therefore, rotation by $\pi/3$ is not a color symmetry and thus the whole coloring is not a perfect coloring. All fundamental regions that are colored with the same color form a subset of D_{12}. The subset that formed with all fundamental regions colored with blue is $\{e, a, a^3, a^4, ba, ba^4\}$, and the subset formed with all fundamental regions colored with white is $\{a^2, a^5, b, ba^2, ba^3, ba^5\}$. Note that neither of the subsets is a subgroup of D_{12}. This example give us a hint that perfect colorings can only be generated by subgroups.

4.3 Perfect Colorings of Pentagon

From the examples shown above and as explained in the sections, we observe that the number of perfect d-colorings is equal to the number of subgroups of D_{2n} of order

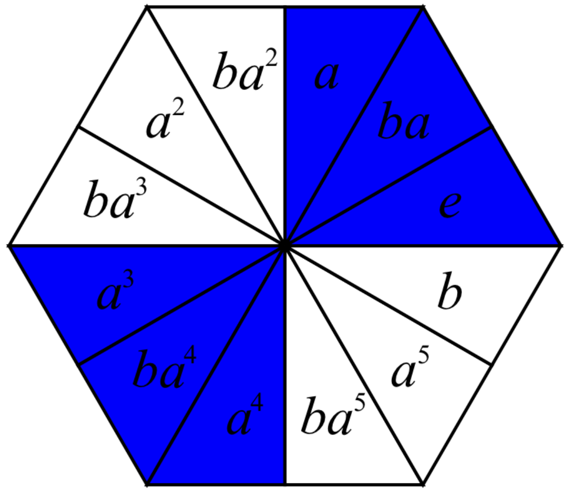


Figure 16: A coloring of hexagon that is not perfect.

$2n/d$. This rule works for the squares and hexagons, whose symmetry groups are D_{2n}, where n is even. From colorings of regular pentagons, we will see the rule can also be applied for designs with symmetry group D_{2n}, where n is odd.

A regular pentagon has 10 symmetries including rotations around the center by, 0, $2\pi/5$, $4\pi/5$, $6\pi/5$, and $8\pi/5$ radians counter-clockwise, and reflections over 5 axes, each through a corner of the pentagon. The set containing the 10 symmetries forms the symmetry group of the pentagon. This group is *the dihedral group* D_{10} *of order 10*.

For D_{10}, let a be rotation counter-clockwise by $2\pi/5$, and b be reflection over the axis shown in Figure 17. Then ba^3 is reflection over the vertical axis, and ba, ba^2, and ba^4 are reflections over other diagonal axes. Also, a^2 is rotation by $4\pi/5$, a^3 is rotation by $6\pi/5$, and a^4 is rotation by $8\pi/5$. Therefore $D_{10} = \langle a, b \mid a^5 = b^2 = e\rangle = \{e, a, a^2, a^3, a^4, b, ba, ba^2, ba^3, ba^4\}$. Figure 18 shows a coordinatized pentagon with elements of D_{10}.

For a pentagon, again, a 1-coloring or 10-coloring is trivial. Each case only has 1 possible equivalent coloring. For 5-colorings, we want to consider subgroups H of D_{10}

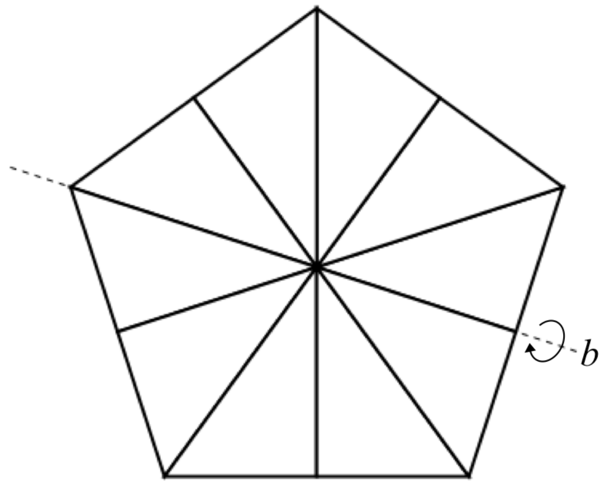


Figure 17: Define b to be reflection over the indicated axis.

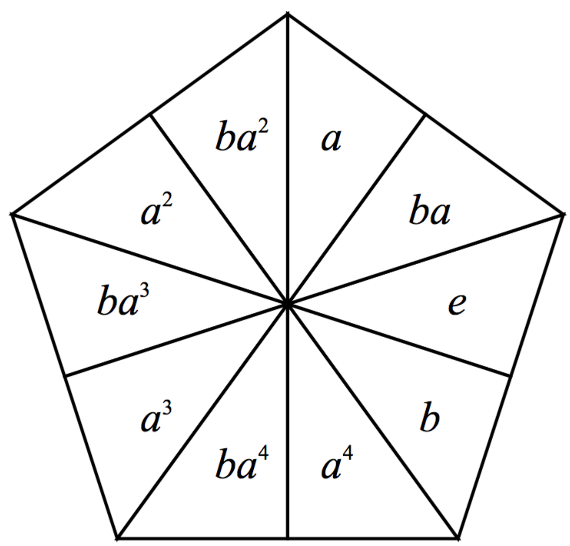


Figure 18: A coordinatized pentagon with elements of D_5.

of order 2. D_{10} has 5 distinct subgroups of order 2 and each yields an inequivalent perfect coloring. The 5 colored pentagons are shown in Figure 19.

D_{10} has only 1 subgroup of order 5, which yields a unique perfect 2-coloring pentagon, which is shown in Figure 20.

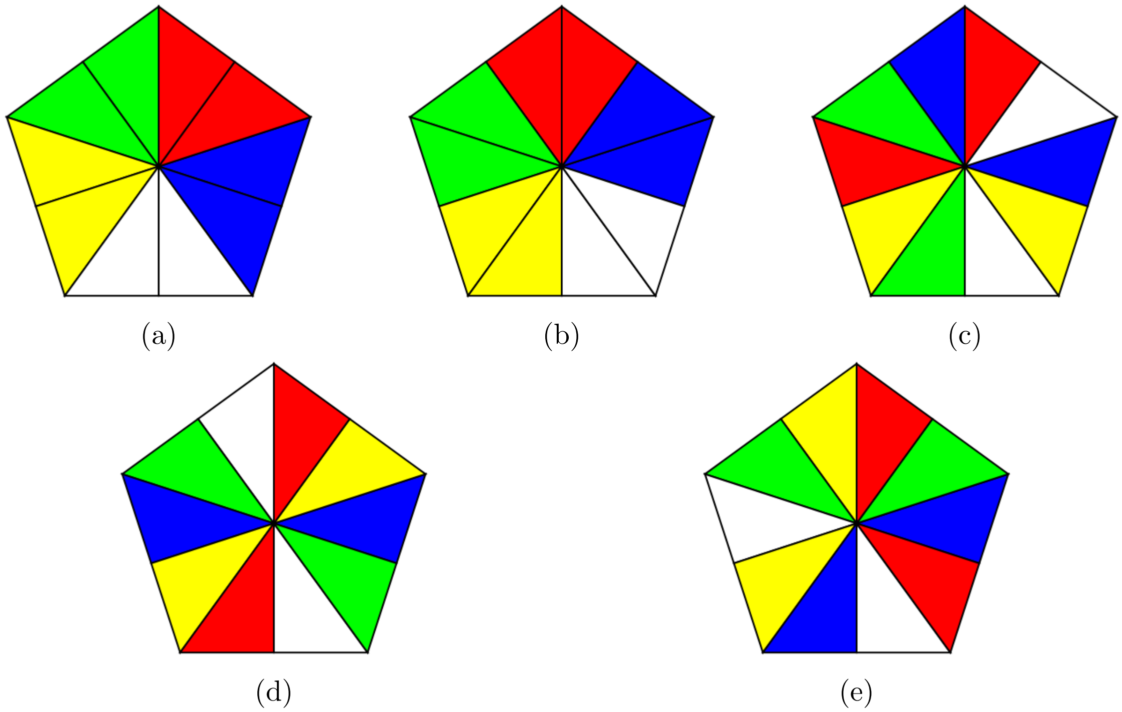


Figure 19: The five 5-colored pentagons. The subgroups used to yield each follows: (a) $H = \{e, b\}$; (b) $H = \{e, ba\}$; (c) $H = \{e, ba^2\}$; (d) $H = \{e, ba^3\}$; and (e) $H = \{e, ba^4\}$.

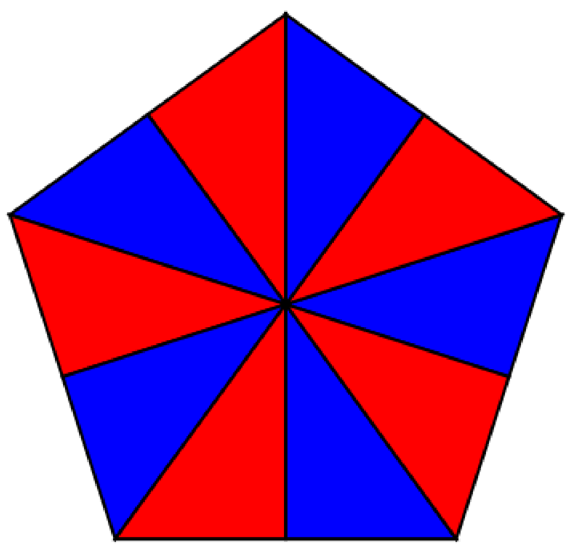


Figure 20: The only 2-colored pentagon. The subgroup used to yield it is $H = \{e, a, a^2, a^3, a^4\}$.

5 Counting Perfect Colorings of a Design With Symmetry Group D_{2n}

As previous examples hint at, there exists a one-to-one correspondence between inequivalent perfect colorings of regular n-gons and distinct subgroups of D_{2n}. In this chapter we generalize this result to any design with symmetry group D_{2n} and prove it. We assume $n \geq 3$ throughout. First, we show the one-to-one correspondence between subgroups of D_{2n} and perfect colorings.

Lemma 5.1 Given a design with symmetry group D_{2n} and a collection of fundamental regions of that design, every perfect coloring is determined by a subgroup of D_{2n}. Also, each distinct subgroup of D_{2n} yields an inequivalent perfect coloring of the design.

Proof: Given a design with symmetry group D_{2n}, fix a beginning region and coordinatize the design by labeling that region with the identity $e \in D_{2n}$. Suppose C is a perfect d-coloring of the design for some $d \mid 2n$. Let c_i be the set of g in D_{2n} such that g is colored with color i. Then $C = c_1 \cup c_2 \cup \cdots \cup c_d$. Without loss of generality, suppose e is colored with color 1. Then $e \in c_1$. Suppose c_1 corresponds to a subset H of D_{2n} such that it is not a subgroup. Denote $H = \{e = g_1, g_2, \cdots, g_{\frac{2n}{d}}\}$; then either there exist some i, $j \in [1, \frac{2n}{d}]$ such that $g_i g_j \notin H$ or there exists some $g \in H$ such that $g^{-1} \notin H$. Suppose the former, that is that there exist some i, j such that $g_i g_j \notin H$. Then $Hg_i = \{g_i, g_2 g_i, \cdots, g_j g_i, \cdots\}$ where $g_i \in H$ and $g_i \in c_1$ but $g_j g_i \notin H$ and $g_j g_i \notin c_1$. Thus, since $g_i \in Hg_i$ but $Hg_i \neq H$, Hg_i is a set of fundamental regions that does not correspond to one of the original c_i's, which is a contradiction. Now, we suppose that there exists some $g \in H$ such that $g^{-1} \notin H$. Then $Hg^{-1} = \{g^{-1}, \cdots, gg^{-1} = e, \cdots\}$ where $e \in H$ and $e \in c_1$ but $g^{-1} \notin H$ and

$g^{-1} \notin c_1$. Similarly, $e \in Hg^{-1}$, but $Hg^{-1} \neq H$, thus Hg^{-1} is a set of fundamental regions that does not correspond to one of the original c_i's. Both cases contradict the assumption of C being a perfect coloring. Therefore, H must be a subgroup of D_{2n} which yields C.

Now, suppose the design has two equivalent perfect colorings C_1 and C_2 yielded from two distinct subgroups H_1 and H_2, respectively, of D_{2n}. Assume that the design is coordinatized in a consistent way, meaning that C_1 and C_2 are coordinatized the same. Clearly, if the two colorings are colored with different numbers of colors, they cannot be transformed to each other by relabeling or applying a symmetry element. Since each distinct color is assigned to one coset of the subgroups, H_1 and H_2 must have the same index in D_{2n}. Then the two subgroups have the same order. Without loss of generality, assume we can transform C_2 to C_1 by applying a symmetry element $g \in D_{2n}$, i.e., $C_1 = C_2 g$. Suppose all regions corresponding to H_1 are colored with color $i \in C_1$. Consider all regions colored with color i in C_2. Those regions correspond to a coset of H_2, say c_i. Then $c_i g = H_1$, and hence is a subgroup of G. Since $c_i g$ is a coset of H_2 and is a group, it must be that $c_i g = H_2$. It follows that $H_2 = H_1$, which contradicts the assumption that H_1 and H_2 are two distinct subgroups. Thus, each distinct subgroup of D_{2n} yields an inequivalent perfect coloring. □

Part (i) of following corollary follows directly from Lemma 5.1. As in Chapter 3, for a subgroup H of D_{2n}, we assign a distinct color to each of the right cosets of H. Then each subgroup of index d gives a perfect d-coloring of the n-gon. Thus, part (ii) of the corollary follows.

Corollary 5.2 For a design with symmetry group D_{2n} and a collection of fundamental regions, if integer d divides $2n$, then

(i) the number of subgroups of D_{2n} is equal to the number of inequivalent perfect colorings;

(ii) the number of subgroups of D_{2n} of index d is equal to the number of inequivalent perfect d-colorings.

Now we investigate further with the number of subgroups of D_{2n}. For $n \in \mathbb{N}$, let $\tau(n)$ denote the number of divisors of n, and $\sigma(n)$ denote the sum of all divisors of n. Then Cavior's Theorem [4] gives us a way to count all subgroups (including itself and the trivial group) of the dihedral group D_{2n}. Cavior's Theorem states that if $n \geq 3$, then the number of subgroups of D_{2n} is $\tau(n) + \sigma(n)$. This theorem leads to the following result.

Theorem 5.3 For any design with symmetry group D_{2n}, there exist $\tau(n) + \sigma(n)$ perfect colorings.

Proof: Given a design with the symmetry group D_{2n} with $n \geq 3$, by Cavior's Theorem, D_{2n} has $\tau(n) + \sigma(n)$ distinct subgroups. Then by Corollary 5.2 (i), the design has $\tau(n) + \sigma(n)$ inequivalent perfect colorings. □

Then the following corollary is an immediate consequence of Theorem 5.3.

Corollary 5.4 For an n-gon, there exist $\tau(n) + \sigma(n)$ perfect colorings.

Proof: When a design is an n-gon, the symmetry group of the design is D_{2n} with $n \geq 3$. Then by Theorem 5.3, the n-gon has $\tau(n) + \sigma(n)$ inequivalent perfect colorings. □

Knowing the number of all perfect colorings may not be enough, it can be more helpful to count perfect d-colorings of a design for each d corresponding to the index of a subgroup, as above. The following theorem provides a way to classify the perfect colorings by the number of colors.

Theorem 5.5 For a design with symmetry group D_{2n}, and any positive divisor d of $2n$,

Case 1: if $d \mid n$ and $\frac{2n}{d} \mid n$, then the number of perfect d-colorings is $d+1$;

Case 2: if $d \mid n$ and $\frac{2n}{d} \nmid n$, then the number of perfect d-colorings is d;

Case 3: if $d \nmid n$ and $\frac{2n}{d} \mid n$, then the number of perfect d-coloring is 1.

Proof: Given a design with its symmetry group $D_{2n} = \langle a, b \mid a^n = b^2 = e \rangle$. Suppose H is a subgroup of D_{2n}. Consider the homomorphism $H \to D_{2n}/\langle a \rangle$. The kernel of this homomorphism is $H \cap \langle a \rangle$. Since $D_{2n}/\langle a \rangle$ is a group of order 2, the homomorphism is either trivial or onto.

If it is a trivial homomorphism, then $H = H \cap \langle a \rangle$, and H is a subgroup of $\langle a \rangle$. The order of the cyclic group $\langle a \rangle$ is n. Then by the Fundamental Theorem of Cyclic Groups, for each positive divisor $\frac{2n}{d}$ of n, $\langle a \rangle$ has exactly 1 subgroup of order $\frac{2n}{d}$. The index of this subgroup is $\frac{2n}{\frac{2n}{d}} = d$. Hence, it yields a perfect d-coloring. Thus, for each d such that $\frac{2n}{d} \mid n$, there is 1 perfect d-coloring.

Suppose the homomorphism $H \to D_{2n}/\langle a \rangle$ is not trivial, then it is onto. Then H has the form $\langle a^d, ba^i \rangle$, where $d \mid n$ and $0 \leq i < d$. Since $\langle a^d, ba^i \rangle = \langle a^d, ba^j \rangle$ if and only if $ba^i = ba^j$ if and only if $i = j \bmod n$, for each $d \mid n$, there are d possible values for i and therefore d such subgroups. Since the kernel of the homomorphism is $H \cap \langle a \rangle = \langle a^d \rangle$, by the First Isomorphism Theorem, $H/\langle a^d \rangle$ is isomorphic to the image of H, which is $D_{2n}/\langle a \rangle$. It follows that $H/\langle a^d \rangle$ has order 2. Then $|H : \langle a^d \rangle| = 2$. Since $|\langle a^d \rangle| = \frac{n}{d}$, $|H| = 2|\langle a^d \rangle| = \frac{2n}{d}$. It follows that the index of the subgroup is d. Thus, for each $d \mid n$, there are d perfect d-colorings.

Therefore, when $d \mid n$ and $\frac{2n}{d} \mid n$, the n-gon has $d+1$ perfect d-colorings, with d of them yielding from the non-cyclic subgroups of D_{2n}, and 1 from a cyclic subgroup. When $d \mid n$ and $\frac{2n}{d} \nmid n$, the n-gon has d perfect d-colorings, yielding from d non-cyclic subgroups of D_{2n}. When $d \nmid n$ and $\frac{2n}{d} \mid n$, the design only has 1 perfect d-coloring, yielding from a cyclic subgroup of of D_{2n}. □

In order to have a better understanding of this theorem, the following example uses hexagons with the symmetry group D_{12} to illustrate the results of this theorem.

Example 5.6 For a hexagon, $n = 6$ and $2n = 12$. Then d can be 12, 6, 4, 3, 2, and 1. When $d = 12$, $d \nmid n$ and $\frac{2n}{d} = 1$ which divides n (Case 3); when $d = 6$, $d \mid n$ and $\frac{2n}{d} = 2$ which divides n (Case 1); when $d = 4$, $d \nmid n$ and $\frac{2n}{d} = 3$ which divides n (Case 3); when $d = 3$, $d \mid n$ and $\frac{2n}{d} = 4$ which does not divide n (Case 2); when $d = 2$, $d \mid n$ and $\frac{2n}{d} = 6$ which divides n (Case 1); and finally when $d = 1$, $d \mid n$ and $\frac{2n}{d} = 12 \nmid n$ (Case 2). Then by Theorem 5.5, the hexagon has 1 perfect 12-coloring, 7 perfect 6-colorings, 1 perfect 4-coloring, 3 perfect 3-colorings, 3 perfect 2-colorings, and 1 perfect 1-coloring. These results are illustrated by the perfect colorings of the hexagon shown in Section 4.2.

When n is odd, we are able to reduce a case and simplify Theorem 5.5, which gives the following corollary.

Corollary 5.7 Let $d \mid 2n$ where n is odd. If d is odd, then there are d perfect d-colorings of the design with symmetry group D_{2n}. If d is even, then there is 1 perfect d-coloring of the design.

Proof: Given a design with symmetry group D_{2n} and n is odd. Let $d \mid 2n$. If d odd, then $d \mid n$ since 2 is not a factor of d. Then $\frac{2n}{d}$ is even, and therefore it does not divide n. Hence, this follows Case 2 from Theorem 5.5 and gives d perfect d-colorings of the design. If d even, then $d \nmid n$ since n is odd. Since 2 is a factor of d, $\frac{2n}{d}$ is odd, and therefore it must divide n. This follows Case 3 and gives 1 perfect d-coloring of the design. □

We can further simplify Theorem 5.5 when n is prime.

Corollary 5.8 If n is prime, then a design with symmetry group D_{2n} has:

1 perfect $2n$-coloring

n perfect n-colorings

1 perfect 2-coloring

1 perfect 1-coloring

Proof: Given a design with symmetry group D_{2n} where n is prime. The only divisors d of $2n$ are $2n$, n, 2 and 1. When $d = 2n$, $\frac{2n}{d} = 1$ which always divides n, but d cannot divide n. This belongs to Case 3. It follows that the design only has 1 perfect $2n$-coloring. When $d = n$, $\frac{2n}{d} = 2$ which does not divide n since we are only looking at $n \geq 3$ and all primes greater than 2 are odd. But since $d \mid n$, we have Case 2 and the design has n perfect n-colorings. When $d = 2$, $\frac{2n}{d} = n$ which divides n, but $d \nmid n$, so this belongs to Case 3. Therefore, the design has 1 perfect 2-coloring. Finally, when $d = 1$ which divides n, $\frac{2n}{d} = 2n$ which does not divide n. This gives us Case 2 and the design has 1 perfect 1-coloring. □

6 Frieze Patterns and Colorings

Thus far, we have only worked with designs with crystallographic groups of rank 0. Recall that crystallographic groups of rank 0 have empty lattices Λ and therefore are finite. The finite symmetry groups of designs are the dihedral groups or its subgroups and only contain rotation and reflection symmetries. In this chapter, we will study designs whose symmetry groups are crystallographic groups G of rank 1. When G has rank 1, its lattice Λ has rank 1. Let $\vec{v}$ be a vector in the Euclidean plane such that $\vec{v} \neq 0$, then as we claimed in Chapter 2, the mapping $T(x) = x + \vec{v}$ is a isometry that acts as a translation, and the point x is translated by distance $||\vec{v}||$ in the direction of $\vec{v}$. The resulting lattice of this translation is $\Lambda = \mathbb{Z}\vec{v}$. We call the vector $\vec{v}$ the *translation vector* of the lattice Λ. The crystallographic group G is made by combining the lattice Λ and a point group $\overline{G}$, as discussed in Chapter 2.

Before we get into any of the crystallographic groups G of rank 1, it is helpful be familiar with the agreed international notation, Hermann-Mauguin notation, for symmetry groups. Compared to the group notations used in previous chapters, the international notations provide better ways to represent groups with translations.[5] First, we use simply the number n to represent the cyclic group C_n, the group of order n that only contains rotations of angle $\frac{2\pi k}{n}$, where k is an integer and $k \in [0, n-1]$. For example, the cyclic group of order 4, C_4, can be denoted by 4. Next, the international notations for the first few dihedral groups are shown in Table 1. The international notations start with numbers that represent the number of rotations in the groups. The following m's stands for mirror lines. For dihedral groups of order $2n$, where n is even, there are two sets of reflection axes. For example, in a square, one set of the axes bisects the angles, and the other set bisects the edges. For all regular n-gons with even n, this is the case. Therefore, all of these dihedral groups have two

m's in their international notations. But for dihedral groups of order $2n$, where n is odd, the two sets of reflection axes coincide. Any reflection axis bisecting an angle also bisects an edge. Thus, these groups only have a single m in their international notations. A summary for the notations of the dihedral groups are shown in Table 2. The international notations can help us understand the common notations used for the symmetry groups that contain translations.

Order	2	4	6	8	10	12	$\cdots$
Group Notation	D_2	D_4	D_6	D_8	D_{10}	D_{12}	$\cdots$
International Notation	$1m$	$2mm$	$3m$	$4mm$	$5m$	$6mm$	$\cdots$

Table 1: Group notations and international notations of the first few dihedral groups.

Order $2n$ when n is	odd	even
Group Notation	D_{2n}	D_{2n}
International Notation	nm	nmm

Table 2: Summary of group notations and international notations of the dihedral groups.

6.1 Frieze Groups and Frieze Patterns

Now we move on to the crystallographic groups of rank 1. Since a crystallographic group of rank 1 is an infinite group, a design whose symmetry group is a crystallographic group of rank 1 is an infinite pattern. The lattice is defined by $\Lambda = \mathbb{Z}\vec{v}$, which indicates that only one direction of translations is involved in crystallographic group of rank 1. This will give us an infinite strip of repeated objects. Conventionally, we choose a horizontal line as the translation axis, i.e., the translation vector $\vec{v}$ is horizontal. These strips are called *frieze patterns.* The repeating object in a frieze pattern is called a *motif.* It is well known that there are only 7 types of frieze patterns.[6] The symmetry group of each frieze pattern is called the *frieze group*.

Since the frieze groups are crystallographic groups of rank 1, they are compositions of a rank 1 lattice Λ with translations defined by $T(x) = x + \vec{v}$, where x represents the motif, and a point group $\overline{G}$. All symmetries in the point group must map the lattice Λ onto itself. Thus a point group can only contain the identity E, rotation R by π, reflection H across a reflection line that is in the direction of the translation vector, and reflection V across a reflection line in the direction that is orthogonal to the translation vector. Note that rotations by another angle other than π are not allowed for a point group because they cannot map the lattice strip onto itself. A point group $\overline{G}$ must be a subgroup of $\{E, R, H, V\}$. Since the order of E is 1 and the orders of R, H, and V are 2, all possible $\overline{G}$ are $\{E\}$, $\{E, R\}$, $\{E, H\}$, $\{E, V\}$, and $\{E, R, H, V\}$. [2]

When a pattern has a symmetry group whose point group $\overline{G} = \{E\}$, the motif of the pattern has no rotation or reflection symmetries and the only symmetries this whole pattern can have are identity and translation. The symmetry group of a such frieze pattern is called $r1$. A frieze pattern whose symmetry group is $r1$ is shown in Figure 21. Each single triangle is a motif and the pattern is simply motifs translated by a fixed distance. Note that the frieze patterns are infinite, so Figure 21 only shows part of the whole pattern. This is the same for all the following frieze patterns.

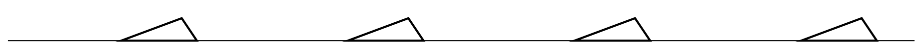


Figure 21: A frieze pattern whose symmetry group is $r1$.

When a pattern has a symmetry group whose point group $\overline{G} = \{E, R\}$, a slightly more complicated frieze group $r2$ is generated. The point group of the frieze group contains identity and rotation by π. The symmetries of the whole frieze group $r2$ are translation, identity, and rotations by π. This forces the motif of a $r2$ frieze pattern to have rotation by π as a symmetry. An example of a frieze pattern with symmetry group $r2$ is shown in Figure 22. The motif of this pattern contains 2 triangles.

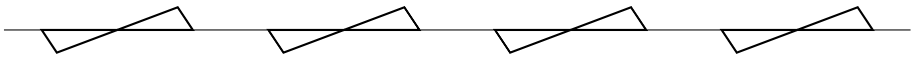


Figure 22: A frieze pattern whose symmetry group is $r2$.

When a pattern has a symmetry group whose point group $\overline{G} = \{E, V\}$, the whole frieze group is called $r1m$. The point group of the frieze group contains identity and reflection across reflection lines that are orthogonal to the translation vector. The symmetries of the whole frieze group $r1m$ are translation, identity, and reflections V. This forces the motif of a $r1m$ frieze pattern to have reflection across a reflection line that is orthogonal to the translation vector as a symmetry. An example of a frieze pattern with symmetry group $r1m$ is shown in Figure 23. The translation vector of this pattern is horizontal, and therefore all the reflection lines are vertical. The motif contains 2 triangles.

Figure 23: A frieze pattern whose symmetry group is $r1m$.

Next, we consider the symmetry groups whose point group $\overline{G} = \{E, H\}$. Recall that the point group $\overline{G}$ is the image of the homomorphism $\phi : G \to O(2)$ defined by $\phi(A) = B$, where $A(x) = Bx + \vec{w}$. This crystallographic group G contains some isometry A such that $\phi(A) = H$. Then $A(x) = Hx + \vec{w}$. We can split the vector $\vec{w}$ into two vectors, with one vector $\vec{w}_1$ parallel to $\vec{v}$ and the other vector $\vec{w}_2$ orthogonal to $\vec{v}$. Then $A(x) = Hx + \vec{w}_1$.[2] This gives us two cases, either $\vec{w}_1 = \vec{0}$ or $\vec{w}_1 \neq \vec{0}$. If $\vec{w}_1 = \vec{0}$, then $A(x) = Hx$. This frieze group G is called $r11m$. The point group of the frieze group contains identity and reflection across reflection line that is parallel to the translation vector. The symmetries of the whole frieze group $r11m$ are translation, identity, and reflections H. This forces the motif of a $r11m$ frieze pattern to have reflection across a reflection line that is parallel to the translation vector as a symmetry. An example of a frieze pattern with symmetry group $r11m$ is shown in Figure 24. The translation vector of this pattern is horizontal, and therefore the reflection line is horizontal. The motif of this pattern contains 2 triangles.

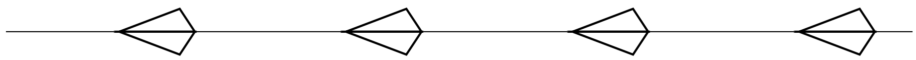


Figure 24: A frieze pattern whose symmetry group is $r11m$.

For the second case in which $\vec{w}_1 \neq \vec{0}$, then $A(x) = Hx + \vec{w}_1$ is called a *glide reflection*. A glide reflection is a combination of a translation and a horizontal reflection. The motif is first reflected across the translation axis, then translated along the translation axis. This frieze group G is called $r11g$. The point group of the frieze group contains identity and a glide reflection. The symmetries of the whole frieze group $r11g$ are translation, identity, and glide reflections. This forces the motif of a $r11g$ frieze pattern to have glide reflection as a symmetry. An example of a frieze pattern with symmetry group $r11g$ is shown in Figure 25. The translation vector of this pattern is horizontal, and therefore the glide reflection line is horizontal. The part contained by the dashed line is the motif of this pattern. The motif of this pattern contains 2 triangles.

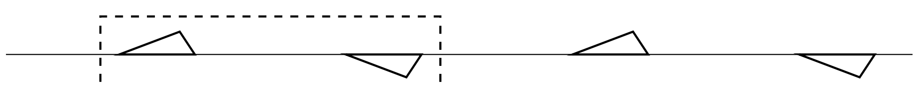


Figure 25: A frieze pattern whose symmetry group is $r11g$.

Similarly, when a pattern has a symmetry group whose point group is $\overline{G} = \{E, R, H, V\}$, there must exists some isometry A that $\phi(A) = H$. Then we also have two cases depending on whether A is a reflection or a glide reflection. When A

is a reflection across the line of the translation vector, we get a frieze group called $r2mm$. The point group of this frieze group contains identity, rotation by π, reflection across reflection lines orthogonal to the translation vector, and reflection across reflection line that is parallel to the translation vector. The symmetries of the whole frieze group $r2mm$ are translation, identity, reflection V and reflection H. This forces the motif of a $r2mm$ frieze pattern to have rotation by π, reflection across a reflection line that is orthogonal to the translation vector, and reflection across a reflection line that is parallel to the translation vector as symmetries. An example of a frieze pattern with symmetry group $r2mm$ is shown in Figure 26. The motif of this pattern contains 4 triangles.

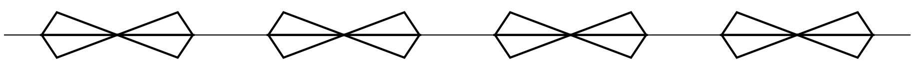


Figure 26: A frieze pattern whose symmetry group is $r2mm$.

When A is a glide reflection, we get a frieze group called $r2mg$. The point group of this frieze group contains identity, rotation by π, reflection across reflection lines orthogonal to the translation vector, and glide reflection. The symmetries of the whole frieze group $r2mm$ are translation, identity, reflection V and glide reflection. This forces the motif of a $r2mg$ frieze pattern to have rotation by π, reflection across a reflection line that is orthogonal to the translation vector, and glide reflection as symmetries. An example frieze pattern with symmetry group $r2mg$ is shown in Figure 27. Again, the part contained by the dashed line is the motif of the $r2mg$ pattern. The motif includes 4 triangles. The center of rotation is at the center of the motif. The glide reflection axis is the same as the translation axis. The vertical reflection axes of this pattern are where two triangles touch.

All of the frieze group notations start with an r. This r stands for repetition along the horizontal axis. The number in the position after the r indicates whether

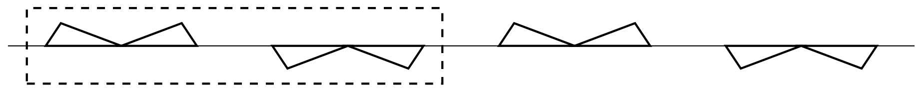


Figure 27: A frieze pattern whose symmetry group is $r2mg$.

the group contains rotation by π. If the rotation symmetry is involved, then the number would be 2, representing C_2; and if the rotation symmetry is not involved in the group, then the number is 1, referring to C_1. The m or 1 in the next position indicates whether there are vertical reflection lines in the pattern. We use m when vertical reflections are present in the group, and 1 if vertical reflections are not present. Similarly, the following position has an m when horizontal reflections are contained in the symmetry group, or a g if the symmetry group includes glide reflections.[6]

We define each of the triangles in the 7 example frieze patterns shown above as a *petal*. Keep in mind that the 7 patterns are examples and the motifs do not need to be formed by triangles. The triangles only give a hint of the way to divide each motif into petals. The motif of a $r1$ frieze pattern contains only 1 petal. The motifs of the $r2$, $r1m$, $r11m$, and $r11g$ frieze patterns all have 2 petals, as in Figure 22, 23, 24, and 25. The motifs of the $r2mm$ and $r2mg$ frieze patterns both have 4 petals, as in Figure 26, and Figure 27. Table 3 summarizes the symmetry elements in the point groups of the 7 frieze groups and number of petals in a motif of each frieze pattern. In order to simplify the descriptions for the elements of the point groups, we assume that the translation vector is horizontal, and we abbreviate "reflection across reflection lines that are parallel to the translation vector" as "horizontal reflection", and "reflection across reflection lines that are orthogonal to the translation vector" as "vertical reflection".

Frieze Group	**Point Group Elements**	**# of Petals**
$r1$	identity (E)	1
$r2$	identity (E), rotation by π (R)	2
$r1m$	identity (E), vertical reflection (V)	2
$r11m$	identity (E), horizontal reflection (H)	2
$r11g$	identity (E), glide reflection (H)	2
$r2mm$	identity (E), rotation by π (R), vertical reflection (V), horizontal reflection (H)	4
$r2mg$	identity (E), rotation by π (R), vertical reflection (V), glide reflection (H)	4

Table 3: Summary of the symmetry elements in the point groups of each frieze group and number of petals in each motif of frieze patterns.

6.2 Colorings of Frieze Patterns

To study the colorings of frieze patterns, we need to start with considering how many fundamental regions each frieze pattern can have. Recall that the fundamental regions of a design must have two properties: (i) The regions are disjoint; (ii) Given any two regions A_i, A_j, there exists a unique symmetry $g \in G$ such that g maps the region A_i onto the region A_j. Because of the property (ii), each petal cannot be subdivided since no symmetry in the frieze groups can map a part of a petal to another part of the same petal. As a result, the motif of $r1$ can have only 1 fundamental region, the motifs of $r2$, $r1m$, $r11m$, and $r11g$ can have 2 fundamental regions, and the motifs of $r2mm$ and $r2mg$ can have 4 fundamental regions.

For a design whose symmetry group is a frieze group, how many ways can we perfect color it? Frieze groups are infinite groups and the lattice of each frieze group is defined as $\Lambda = \mathbb{Z}\vec{v}$. Then if we allow an infinite number of colors, then there are infinitely many ways that we can color an infinite strip. For example, we can assign a distinct color to each motif, then we have a coloring with infinitely many colors. Then this coloring is isomorphic to $\mathbb{Z}$. If we color the whole strip with a single color, then the coloring is isomorphic to $\mathbb{Z}_1$. If we color the whole strip with two colors, for example, color a motif blue, color the adjacent motif red, color the following one blue, and infinitely alternating colors, then the end result coloring is isomorphic to $\mathbb{Z}_2$. Generally, if we perfect color the whole frieze pattern with n colors, then the coloring is isomorphic to $\mathbb{Z}_n$.

Now we limit the question to study how many ways can we color an individual motif. Then we only use the subgroups of the point group of each frieze group. Since the motif of $r1$ only has one fundamental region, it only has the trivial 1-coloring. This makes sense because the point group of $r1$ frieze group only contains the identity. For $r2$, $r1m$, $r11m$, and $r11g$, the motifs have two fundamental regions. Then we can either color the two fundamental regions with the same color or two different colors. Therefore, each of these three frieze patterns has the trivial 1-coloring and a 2-coloring. This also agrees with the fact that the point groups of the $r2$, $r1m$, $r11m$ and $r11g$ frieze groups all have 2 elements, and therefore each of the three point groups only have two subgroups, itself and the trivial group. Since the motif of $r2mm$ has 4 fundamental regions, it can have the 1-coloring, three 2-colorings, and a 4-coloring. The 1-coloring and 4-coloring are trivial. We either color the whole motif using a single color, or color each fundamental region using a different color. The three 2-colorings of $r2mm$ are slightly more complicated, but they can be determined using the subgroup scheme as previously. The 2-colorings and their corresponding subgroups of the point group of $r2mm$ are shown in Figure 28.

Similar to $r2mm$, the motif of $r2mg$ also has 4 fundamental regions. The $r2mg$ frieze pattern also has a trivial 1-coloring, three 2-colorings and a 4-coloring. The three 2-colorings of $r2mg$ are shown in Figure 29.

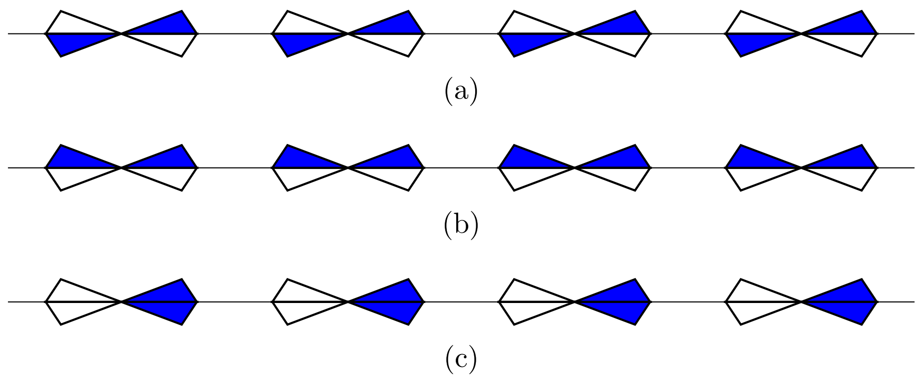


Figure 28: The three 2-colorings of the $r2mm$ frieze pattern. The subgroups of the point group of $r2mm$ frieze group used to yield each follows: (a) $H = \{\text{identity, rotation by } \pi\}$; (b) $H = \{\text{identity, vertical reflection}\}$; and (c) $H = \{\text{identity, horizontal reflection}\}$.

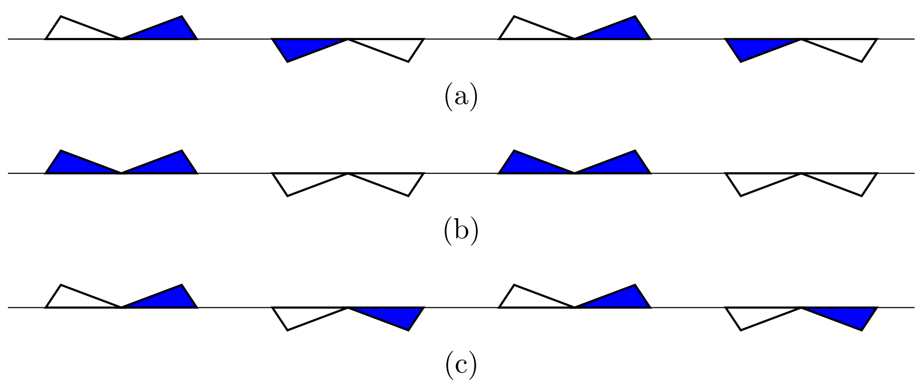


Figure 29: The three 2-colorings of the $r2mg$ frieze pattern. The subgroups of the point group of $r2mg$ frieze group used to yield each follows: (a) $H = \{\text{identity, rotation by } \pi\}$; (b) $H = \{\text{identity, vertical reflection}\}$; and (c) $H = \{\text{identity, glide reflection}\}$.

6.3 Imposing D_{2n} Symmetry on Motif of a Colored Frieze Pattern

In this section, we impose the D_{2n} symmetry on the motif of each frieze pattern. Let's assume that each frieze pattern is already colored corresponding to the frieze group symmetries. Using the same coordinatization system as previously, let the fundamental region e be a petal of the motifs. Then for the $r1$ frieze patterns, the e fundamental regions are the only petals. For the $r2$ frieze group, the symmetry element other than translations is rotation by π. Since the fundamental region e is a petal, the other petal can be determined by rotating e by π, which gives the fundamental region $a^{\frac{n}{2}} \in D_{2n}$. Therefore the two petals in $r2$ frieze patterns are fundamental regions corresponding to e and $a^{\frac{n}{2}}$. For the $r1m$ frieze group, the symmetry elements are translations and vertical reflections. Again, the identity fundamental region e is a petal and the other petal can be obtained by reflecting e by the vertical reflection axis, which gives $ba^{\frac{n}{2}} \in D_{2n}$. Then the two petals in $r1m$ frieze patterns are fundamental regions that corresponds to e and $ba^{\frac{n}{2}}$. Similarly, it can be determined that the two petals in $r11m$ are fundamental regions that corresponds to e and $b \in D_{2n}$, and the four petals in $r2mm$ are fundamental regions corresponding to e, $ba^{\frac{n}{2}}$, $a^{\frac{n}{2}}$, and $b \in D_{2n}$. For the other two frieze groups, glide reflections are involved. Since glide reflections are not applicable to designs with finite symmetry groups, the $r11g$ and $r2mg$ frieze groups are not discussed in the remainder of this chapter. Note that we are only considering D_{2n} where n is even; otherwise, elements such as $a^{\frac{n}{2}}$ do not make sense.

Take an infinite strip of squares as an example. The pattern is shown in Figure 30. As mentioned before, the symmetry group of a square is D_8. The petal of $r1$ is the fundamental region $e \in D_8$. The two petals in $r2$ are e and $a^2 \in D_8$. The two petals in $r1m$ are e and $ba^2 \in D_8$. The two petals in $r11m$ are e and $b \in D_8$. The four petals in $r2mm$ are e, ba^2, a^2, and $b \in D_8$.

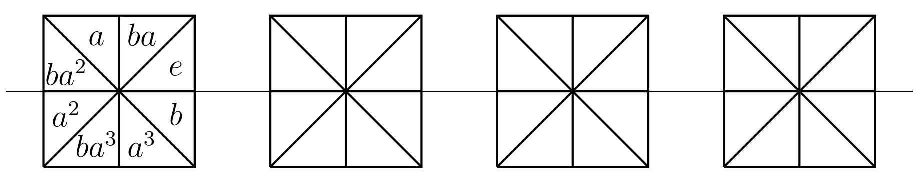


Figure 30: An infinite strip of squares. The square is labeled using the same coordinatization system as previously.

The method of determining perfect colorings of the motif is based on counting subgroups of D_{2n} containing certain elements. For the motif of $r1$, since it only has one fundamental region e, we count how many subgroups of D_{2n} contains the element e. Since e is contained in every subgroup, all perfect colorings of an n-gon can be used to color the $r1$ frieze pattern, assuming the petal of the motif is colored.

For the motif of $r2$, if the two petals are 1-colored, then the number of perfect colorings of an n-gon that can be used to color the whole motif is equal to the number of subgroups of D_{2n} that contain the element $a^{\frac{n}{2}}$. If the two petals are 2-colored, then the number of perfect colorings of an n-gon that can be used to color the whole motif is equal to the number of subgroups of D_{2n} that do not contain the element $a^{\frac{n}{2}}$.

Take the infinite strip of squares as an example, if the pattern is 1-colored according to the $r2$ frieze group, then the pattern will look like Figure 31. Then we refer back to perfect colorings shown in Section 4.1 to determine which perfect colorings of a square can fit in the motifs of the colored pattern. All perfect colorings that fit in the 1-colored frieze pattern are shown in Figure 32. If we look closely at the subgroups of D_8 that yield these colorings, we can see that all the subgroups contain the elements e and a^2. Therefore when we determine whether a perfect coloring can fit in the motifs of a frieze pattern that is 1-colored according to $r2$, we need to check if the subgroup corresponding to the coloring contains e and a^2. Since e is contained in any subgroup, all we have to check is whether the subgroup contains a^2.

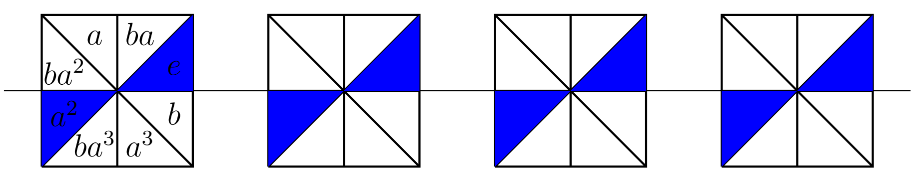
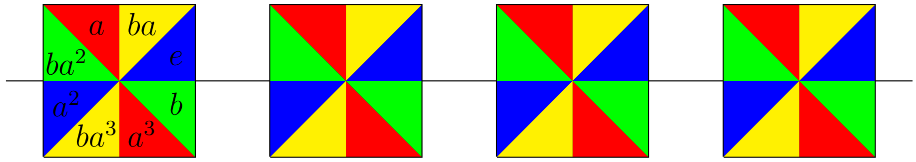
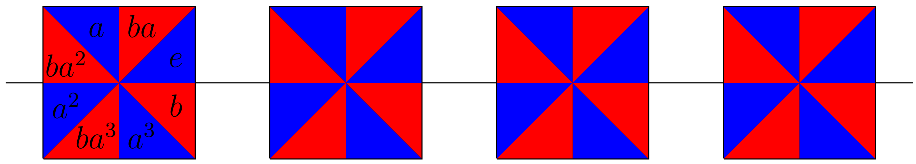


Figure 31: An infinite strip of squares. The pattern is 1-colored using the $r2$ frieze group.

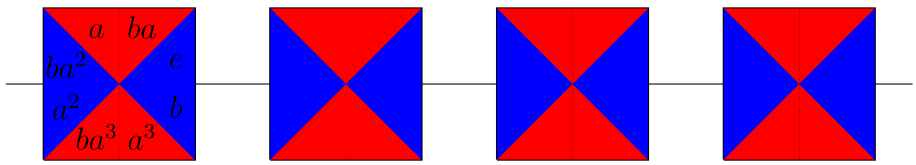
If the infinite strip of squares pattern is 2-colored using the $r2$ frieze group, then



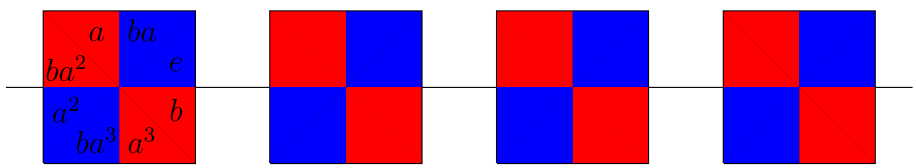
(a)



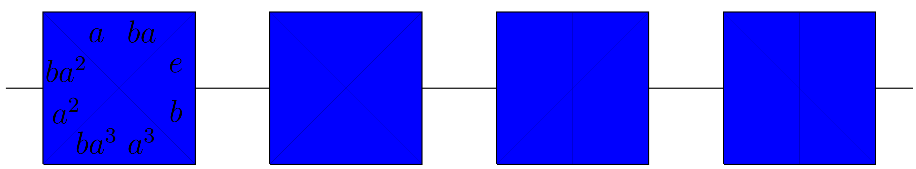
(b)



(c)



(d)



(e)

Figure 32: The infinite strips of squares are 1-colored corresponding to $r2$. The subgroups used to yield each follows: (a) $H = \{e, a^2\}$; (b) $H = \{e, a, a^2, a^3\}$; (c) $H = \{e, a^2, b, ba^2\}$; (d) $H = \{e, a^2, ba, ba^3\}$; (e) $H = D_8$.

the pattern will look like Figure 33. Then we know that the elements e and a^2 cannot be in the same coset, since each coset is assigned a distinct color. The coset that contains the element e must be a subgroup of D_8. Therefore, all subgroups of D_8 that generate perfect colorings which can be used to color the whole square cannot contain the element a^2.

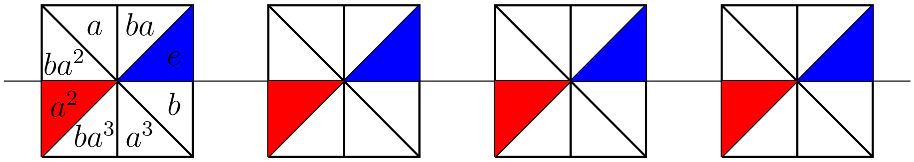


Figure 33: An infinite strip of squares. The pattern is 2-colored using the $r2$ frieze group.

Let's consider the $r1m$ and $r11m$ frieze patterns before returning to $r2$ later in this section. The results for $r1m$ and $r11m$ are simpler than the results for $r2$. For the motif of a $r1m$ frieze pattern, if the two petals are 1-colored, then the number of perfect colorings of an n-gon that can be used to color the whole motif is equal to the number of subgroups of D_{2n} that contain the element $ba^{\frac{n}{2}}$. If the two petals are 2-colored, then the number of perfect colorings of an n-gon that can be used to color the whole motif is equal to the number of subgroups of D_{2n} that do not contain the element $ba^{\frac{n}{2}}$. Similarly, for a $r11m$ pattern, the perfect colorings of an n-gon that can be used to color the whole motif when the two petals are 1-colored or 2-colored is corresponding to subgroups of D_{2n} that do contain or do not contain the element b. The following lemma tells us the number of subgroups of D_{2n} that contains elements b or $ba^{\frac{n}{2}}$.

Lemma 6.1 For D_{2n}, where n is even, the number of subgroups containing the element ba^k, $0 \leq k \leq n-1$, is $\tau(n)$.

Proof: Given an n-gon, where n is even, its symmetry group is $D_{2n} = \langle a, b \mid a^n = b^2 = e \rangle$. Let H be a subgroup of D_{2n} such that the element $ba^k \in H$ for any $0 \leq k \leq n-1$. H is either a cyclic group or a non-cyclic group.

If H is cyclic, then H is a subgroup of $\langle a \rangle$. Then the element $ba^k \notin H$, which contradicts our assumption. Therefore H is not a cyclic subgroup. As in the proof of Theorem 5.5, H has the form $\langle a^d, ba^i \rangle$, where $d \mid n$ and $0 \leq i < d$, and $\langle a^d, ba^i \rangle = \langle a^d, ba^j \rangle$ if and only if $ba^i = ba^j$ if and only if $i = j \mod n$. So for each $d \mid n$, there exists exactly 1 distinct subgroup of D_{2n} that contains the element ba^k. It follows that the number of subgroups containing ba^k is equal to the number of d that divides n, and is equal to the number of divisors of n, which is $\tau(n)$. $\square$

Theorem 6.2 follows directly from this lemma and Corollary 5.4.

Theorem 6.2 Let the motif of a $r1m$ or $r11m$ frieze pattern have D_{2n} symmetry. If the two petals of the motif are 1-colored, i.e., the fundamental region $ba^{\frac{n}{2}}$ or b is colored with the same color as the fundamental region e, then there are $\tau(n)$ perfect colorings of the n-gon that can be used to color the whole motif. If the two petals are 2-colored, i.e., the fundamental region $ba^{\frac{n}{2}}$ or b is colored with a different color from the color of the fundamental region e, then there are $\sigma(n)$ perfect colorings of the n-gon that can be used to color the whole motif.

Proof: If a pattern is 1-colored according to the $r1m$ group and its motif has D_{2n} symmetry, then a perfect coloring of an n-gon can be used to color the whole motif if and only if its corresponding subgroup of D_{2n} contains the elements e and $ba^{\frac{n}{2}}$. By Lemma 6.1, there are $\tau(n)$ subgroups of D_{2n} that contain $ba^{\frac{n}{2}}$. Then there are $\tau(n)$ colorings that can be used to color the whole motif.

If the pattern is 2-colored according to the $r1m$ group and its motif has D_{2n} symmetry, then a perfect coloring of an n-gon can to be used to color the whole motif if and only if its corresponding subgroup of D_{2n} does not contain the element $ba^{\frac{n}{2}}$. By Corollary 5.4, for any n-gon, there exists $\tau(n)+\sigma(n)$ perfect colorings. We know that $\tau(n)$ subgroups of D_{2n} contain $ba^{\frac{n}{2}}$. It follows that there are $\tau(n)+\sigma(n)-\tau(n) = \sigma(n)$ subgroups do not contain the element $ba^{\frac{n}{2}}$. Hence, there are $\sigma(n)$ perfect colorings of an n-gon that can be used to color the whole motif.

Similarly, if a pattern is 1-colored according to the $r11m$ group, and its motif has D_{2n} symmetry, then a perfect coloring of an n-gon can to be used to color the whole motif if and only if its corresponding subgroup of D_{2n} contains the elements e and b. Again, since all subgroups contains e, we only need to find subgroups that contains b. Again, by Lemma 6.1, there are $\tau(n)$ subgroups of D_{2n} that contain b. It follows that there are $\tau(n)$ colorings that can be used to color the whole motif.

If the pattern is 2-colored according to the $r11m$ group, and its motif has D_{2n} symmetry, then a perfect coloring of an n-gon can to be used to color the whole motif if and only if its corresponding subgroup of D_{2n} does not contain the element b. As above, there are $\sigma(n)$ subgroups of D_{2n} not containing the element b. Thus, there are $\sigma(n)$ perfect colorings of an n-gon that can be used to color the whole motif. □

We can illustrate this theorem with the example of the infinite strip of squares. To demonstrate Theorem 6.2, the motif 1-colored corresponding to $r1m$ and $r11m$ are shown in Figure 34. Now we refer back to perfect colorings of squares shown in Figure 9 and Figure 10. For the pattern 1-colored corresponding to $r1m$, Figure 9 (d), Figure 10 (b), and the trivial 1-coloring can be used to color the whole square. For the pattern 1-colored corresponding to $r11m$, Figure 9 (b), Figure 10 (b), and the trivial 1-coloring can be used to color the whole squares. For both cases, there are 3 perfect colorings of a square that can be used to color the whole square motif. By Theorem 6.2, there should be $\tau(4)$(number of divisors of 4) perfect colorings, and $\tau(4) = 3$. (1, 2, and 4 are the divisors of 4.) If the motifs are 2-colored instead of 1-colored, then for each case, all other perfect colorings can be used to color the whole motif. Since there are 10 total perfect colorings of a square, we have $10-3 = 7$ perfect coloring for each case. By Theorem 6.2, there should be $\sigma(4)$ (sum of all divisors of

4) perfect colorings, and $\sigma(4) = 7$. $(1 + 2 + 4 = 7.)$ Our previous results illustrate Theorem 6.2.

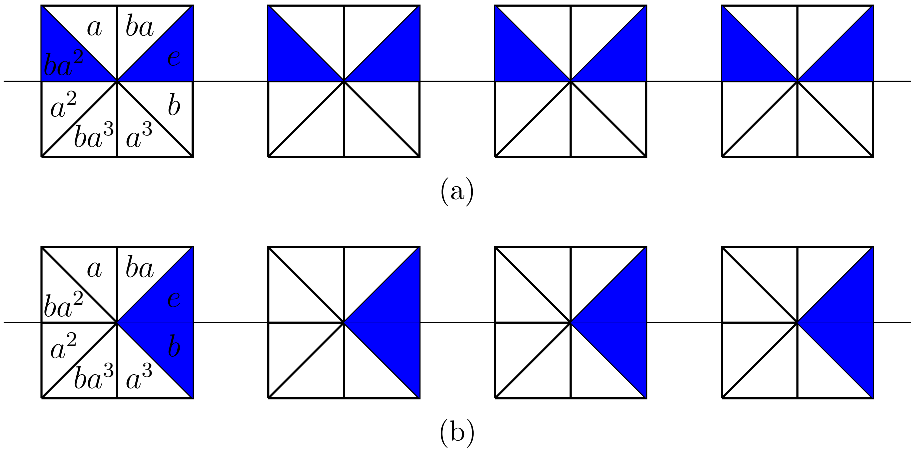


Figure 34: The infinite strips of squares are 1-colored corresponding to (a) $r1m$ and (b) $r11m$.

Now we return to the $r2$ frieze patterns. Counting subgroups of D_{2n} that contain the element $a^{\frac{n}{2}}$ is much more complicated, because the element $a^{\frac{n}{2}}$ is contained in both cyclic and non-cyclic subgroups. Though we do not have a general result, the following theorems work for special cases of n.

Lemma 6.3 For D_{2n}, where $n = 2p$ and p is prime, there are $p + 3$ subgroups containing the element $a^{\frac{n}{2}}$. To be more specific:

1 of the subgroups has order 2,

p of the subgroups have order 4,

1 of the subgroups has order n,

1 of the subgroups has order $2n$.

Proof: Given D_{2n} for some $n = 2p$ and p is prime, p is either 2 or odd. First, we consider the case where $p = 2$. When p is 2, $n = 4$ and we have the dihedral group of order 8. It is easy to list all subgroups of D_8 that contain the element a^2. We have only one subgroup $\{e, a^2\}$ of order 2, $p + 1 = 3$ subgroups of order $n = 4$, which are $\langle a^2, b\rangle$, $\langle a^2, ba\rangle$, and $\langle a\rangle$, and finally one subgroup of order $2n = 8$, which is the D_8 itself.

Then we consider the case where p is an odd prime. The possible orders of a subgroup of D_{2n}, where $n = 2p$ and p is an odd prime, are 1, 2, 4, p, $n = 2p$, and $2n = 4p$. For an element to be contained in a subgroup, the order of the element must divide the order of the subgroup. Since the order of $a^{\frac{n}{2}}$ is 2, we can eliminate some of the possible orders. Clearly, 1 cannot be divided by 2. Since $p \neq 2$, then p is not even and therefore cannot be divided by 2, so the order of a subgroup cannot be p. Then all the possible orders of a subgroup that contains $a^{\frac{n}{2}} \in D_{2n}$ are 2, 4, n, and $2n$.

- When the subgroup has order 2, the only subgroup that contains $a^{\frac{n}{2}}$ is $\{e, a^{\frac{n}{2}}\}$. No other subgroups of order 2 can contain $a^{\frac{n}{2}}$.

- When the subgroup has order 4, the subgroup cannot be $\langle a^{\frac{n}{4}}\rangle$ since p is odd and therefore the element $a^{\frac{n}{4}} = a^{\frac{p}{2}}$ does not exist. Then the subgroup must have the structure $\langle a^{\frac{n}{2}}, ba^i\rangle = \{e, a^{\frac{n}{2}}, ba^i, ba^{i+\frac{n}{2}}\}$, and $ba^i = ba^j$ if and only if $i = j \mod \frac{n}{2}$. Since $\frac{n}{2} = p$, there are p distinct subgroups of order 4. The element $a^{\frac{n}{2}}$ is contained in all such subgroups.

- When the subgroup has order n, the only subgroup that contains the element $a^{\frac{n}{2}}$ is the cyclic subgroup $\langle a\rangle$.

- When the subgroup has order $2n$, the subgroup must be D_{2n} itself, which contains the element $a^{\frac{n}{2}}$.

Thus, there are $1 + p + 1 + 1 = p + 3$ total subgroups that contain the element $a^{\frac{n}{2}}$. $\square$

Theorem 6.4 Let the motif of $r2$ have D_{2n} symmetry where $n = 2p$ and p is prime. If the two petals of the motif are 1-colored, i.e., the fundamental region $a^{\frac{n}{2}}$ is colored with the same color as the fundamental region e, then there are $p+3$ perfect colorings of the n-gon that can be used to color the whole motif. To be more specific:

1 of the perfect colorings is an n-coloring,

p of the perfect colorings are p-colorings,

1 of the perfect colorings is a 2-coloring,

1 of the perfect colorings is a trivial 1-coloring.

Proof: If a pattern is 1-colored according to the $r2$ group and its motif has D_{2n} symmetry, then a perfect coloring of an n-gon can to be used to color the whole motif if and only if its corresponding subgroup of D_{2n} contains the elements e and $a^{\frac{n}{2}}$. Since all subgroup contains e, we only need subgroups that contain $a^{\frac{n}{2}}$. Also, the number of colors of a coloring is equal to the index of the corresponding subgroup, which is equal to the group order divided by the subgroup order. Then by Lemma 6.3, when $n = 2p$ and p is prime, there exist 1 subgroup of D_{2n} that has order 2 that contains $a^{\frac{n}{2}}$, which yields 1 perfect n-coloring. The p subgroups of order 4 yield p perfect colorings with $2n/4 = \frac{n}{2} = p$ colors. The subgroup of order n yields a perfect 2-coloring, and the subgroup of order $2n$ yields a trivial 1-coloring. Thus, there exist $1 + p + 1 + 1 = p + 3$ perfect colorings of the n-gon that can be used to color the whole motif. □

Since $4 = 2 \times 2$ and 2 is prime, we can again use the infinite strip of squares example to illustrate Theorem 6.4. The 1-colored $r2$ frieze pattern of squares are shown in Figure 31. The perfect colorings of a square that can be used to color the whole motif are Figure 9 (a), which is a 4-coloring, all 3 colorings in Figure 10, and the trivial 1-coloring. The end results are shown in Figure 32. By Theorem 6.4, the whole motif can be colored using 1 perfect 4-coloring, $p = 2$ perfect $\frac{4}{2}$-colorings, 1 perfect 2-coloring, and 1 trivial 1-coloring. Since $\frac{4}{2} = 2$, we actually should have 3 perfect 2-colorings. These results thus illustrate Theorem 6.4.

Lemma 6.5 Given D_{2n}, where $n = 2m$ and m is odd, then for any positive divisor d of $2n$,

Case 1: if $d \mid n$ and $\frac{2n}{d} \mid n$, then there is 1 subgroup of D_{2n} of index d that contains the element $a^{\frac{n}{2}}$;

Case 2: if $d \mid n$ and $\frac{2n}{d} \nmid n$, then there are d subgroups of D_{2n} of index d that contain the element $a^{\frac{n}{2}}$.

Proof: Let H be a subgroup of D_{2n} of index d. Then as shown in proof of Theorem 5.5, H is either a cyclic subgroup of $\langle a \rangle$, or a non-cyclic subgroup of the form $\langle a^d, ba^i \rangle$. The order of H is $\frac{2n}{d}$.

> Case 1: If $d \mid n$ and $\frac{2n}{d} \mid n$, then by Theorem 5.5, there exists 1 cyclic subgroup of D_{2n} of index d and d non-cyclic subgroups of D_{2n} of index d. First, we assume that H is a cyclic subgroup of $\langle a \rangle$. Since the order of the element $a^{\frac{n}{2}}$ is 2, and $a^{\frac{n}{2}} \in \langle a \rangle$, the element $a^{\frac{n}{2}} \in H$ if and only if $2 \mid |H|$ where $|H| = \frac{2n}{d}$. We have $d \mid n$, then $2 \mid \frac{2n}{d}$, it follows that $a^{\frac{n}{2}} \in H$. Next, we assume that H is a non-cyclic subgroup of D_{2n}. Then $H = \langle a^d, ba^i \rangle$ for some i that $0 \leq i < d$. The element $a^{\frac{n}{2}} = a^m \in H$ if and only if $a^m \in \langle a^d \rangle$ if and only if $|a^m| \mid |\langle a^d \rangle|$ if and only if $2 \mid \frac{n}{d}$. Since we have $\frac{2n}{d} \mid n$ and $n = 2m$, $\frac{n}{d} \mid m$. Since m is odd and $n = 2m$ is even, d must be even. It follows that $\frac{n}{d} = \frac{2m}{d}$ is odd. Since $2 \nmid \frac{n}{d}$, the element $a^{\frac{n}{2}} \notin H$. Thus when $d \mid n$ and $\frac{2n}{d} \mid n$, there exists only a cyclic subgroup of D_{2n} of index d that contains the element $a^{\frac{n}{2}}$.

Case 2: If $d \mid n$ and $\frac{2n}{d} \nmid n$, then by Theorem 5.5, there exists d non-cyclic subgroups of D_{2n} of index d. Let H be one of the subgroups, then as shown above, $H = \langle a^d, ba^i \rangle$ and the element $a^{\frac{n}{2}} \in H$ if and only if the order of $a^{\frac{n}{2}}$ divides the order of the subgroup $\langle a^d \rangle$, which is $2 \mid \frac{n}{d}$. Since $d \mid n$, there exists some integer i such that $n = di$. We have $\frac{2n}{d} \nmid n$, then $\frac{2di}{d} \nmid di$, and therefore $2i \nmid di$. It follows that $2 \nmid d$ and d is odd. Since m is odd, $\frac{n}{d} = \frac{2m}{d}$ is even. Hence $2 \mid \frac{n}{d}$, and the element $a^{\frac{n}{2}} \in H$. Thus all d non-cyclic subgroups of D_{2n} of index d contain the element $a^{\frac{n}{2}}$. □

Theorem 6.6 Consider the motif of $r2$ with D_{2n} symmetry where $n = 2m$ and m is odd. Let the two petals of the motif be 1-colored, i.e., the fundamental region $a^{\frac{n}{2}}$ is colored with the same color as the fundamental region e. For any positive divisor d of $2n$,

Case 1: if $d \mid n$ and $\frac{2n}{d} \mid n$, then there is 1 perfect d-coloring of an n-gon that can be used to color the whole motif;

Case 2: if $d \mid n$ and $\frac{2n}{d} \nmid n$, then there are d perfect d-colorings of an n-gon that can be used to color the whole motif.

Proof: If a pattern is 1-colored according to the $r2$ frieze group and its motif has D_{2n} symmetry, then a perfect coloring of an n-gon can to be used to color the whole motif if and only if its corresponding subgroup of D_{2n} contains the elements e and $a^{\frac{n}{2}}$. Since all subgroups contain e, we only need subgroups that contain $a^{\frac{n}{2}}$. Then by Lemma 6.5, for any positive divisor d of $2n$, if $d \mid n$ and $\frac{2n}{d} \mid n$, then there exist 1 subgroup of D_{2n} of index d that contains the element $a^{\frac{n}{2}}$, which yields 1 perfect d-coloring of an n-gon that can be used to color the whole motif. When $d \mid n$ and $\frac{2n}{d} \nmid n$, there exists d subgroups of D_{2n} of index d that contain the element $a^{\frac{n}{2}}$, which yield d perfect d-colorings of an n-gon that can be used to color the whole motif. □

When the petals of the motif of $r2$ with D_{2n} symmetry are 2-colored, we can find the number of perfect colorings of an n-gon we can use to color the whole motif by subtracting the number of perfect colorings we can use to color the motif when the petals are 1-colored from the total number of perfect colorings.

7 Wallpaper Patterns and Colorings

We have discussed crystallographic groups of rank 0 and 1 in the previous chapters. In this chapter, we introduce crystallographic groups of rank 2, the last type of 2-dimensional Euclidean crystallographic groups. We also study the colorings of a design whose symmetry group is a crystallographic group of rank 2.

7.1 Wallpaper Groups and Wallpaper Patterns

When a crystallographic group G has rank 2, its lattice Λ has rank 2. Then the lattice is defined as $\Lambda = \mathbb{Z}\vec{v}_1 + \mathbb{Z}\vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are two linearly independent vectors in $\mathbb{R}^2$. In a frieze pattern, the motif is only translated along one axis and forms a strip. Clearly, in two-dimensions, objects can be moved in more than one direction. When we move an object along two independent directions, $\vec{v}_1$ and $\vec{v}_2$, we can obtain a repeating pattern that spans the entire 2-dimensional plane. We call such a pattern a *wallpaper pattern*. Figure 35 is a very basic example of a wallpaper pattern. The points in Figure 35 are motifs of this pattern. The two vectors $\vec{v}_1$ and $\vec{v}_2$ indicated in the figure are the two *translation vectors* of this wallpaper pattern. The lines in the directions of the two translation vectors are referred to as *translation axes*. The parallelogram formed with the two dashed lines and $\vec{v}_1$ and $\vec{v}_2$ is called the *unit cell* of this pattern.

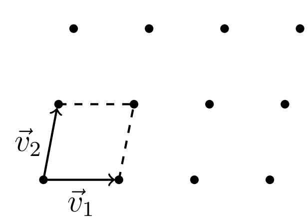


Figure 35: A wallpaper pattern.

The unit cell of a wallpaper pattern will always be a parallelogram. As the angle between the two translation vectors, denoted by α, and the length of the two translation vectors change, the unit cell may be different. It is well known that there are 5 types of unit cells depending on its shape.[6] A summary of the 5 types of unit cells and their requirements for the angle between translation vectors and length of translation vectors are shown in Table 4.

Unit Cell Type	α	**Length of $\vec{v}_1$ and $\vec{v}_2$**
Parallelogram	/	/
Rectangular	$\alpha = 90°$	/
Rhombic	$\alpha \neq 90°$	$\|\|\vec{v}_1\|\| = \|\|\vec{v}_2\|\|$
Square	$\alpha = 90°$	$\|\|\vec{v}_1\|\| = \|\|\vec{v}_2\|\|$
Hexagonal	$\alpha = 60°$	$\|\|\vec{v}_1\|\| = \|\|\vec{v}_2\|\|$

Table 4: Summary of the five unit cell types and their requirements for the angle between translation vectors and length of translation vectors.

It is well known that there are 17 types of wallpaper patterns.[6] The symmetry group of each wallpaper pattern is called the *wallpaper group*. Since there are more translation axes involved in wallpaper patterns compared to frieze patterns, the symmetry elements in wallpaper groups are much more complicated. Before we derive into the patterns, in order to better way to distinguish between different wallpaper patterns, it is necessary to clarify or define some terms for symmetry operations. First, if a motif in a wallpaper pattern only has reflection across one of its translation axes, then we will refer to this reflection as *single-axis reflection*. An example of a wallpaper pattern with single-axis reflection is shown in Figure 36 (a). Some motifs of a wallpaper have reflections across both translation axes. Usually, the two sets of reflection axes are horizontal axes and vertical axes; but in order to be more generic, the two reflections will be referred to as *1-axis reflection* and *2-axis reflection*. An example of a wallpaper pattern with two sets of reflection axes is shown in Figure 36 (b). When the unit cell is square or hexagonal, more reflections are involved, and we will discuss them in detail later in this chapter. We then need to separate rotations according to their centers of rotation. The rotations whose centers are on reflection axes will be referred to as rotations, and the rotations whose centers are not on reflection axes will be referred to as *off-axis rotations*, which is denoted by *o-rotations*. An example of a wallpaper pattern with rotations that are not off-axis is shown in Figure 37 (a). The example of a wallpaper pattern with off-axis rotations is shown in Figure 37 (b). Note that for Figure 37 (b), even if it contains o-rotations, it can also have rotation centers that are on reflection axes, and such rotation centers are at the center of each motif.

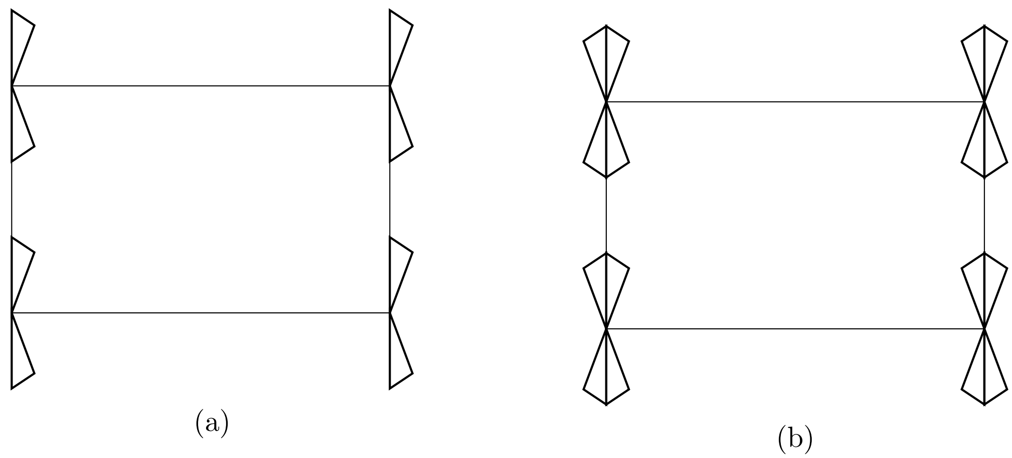


Figure 36: (a) The motif in this pattern only has single-axis reflections (horizontal axes); (b) The motif in this pattern has two sets of reflection axes (horizontal axes and vertical axes), which are referred as 1-axis reflections and 2-axis reflections.

It is well known that the only rotations occurring in a motif of a wallpaper pattern are of angles $\frac{2\pi k}{n}$, where $n = 2, 3, 4,$ or 6, and k is an integer such that $k \in [0, n-1]$.[6] Since for any crystallographic group G, all symmetries in the point group $\overline{G}$ must map the lattice Λ onto itself, the point group of a crystallographic group of rank 2 must be one of the following: $C_1, C_2, C_3, C_4, C_6, D_2, D_4, D_6, D_8,$ or D_{12}. [2]

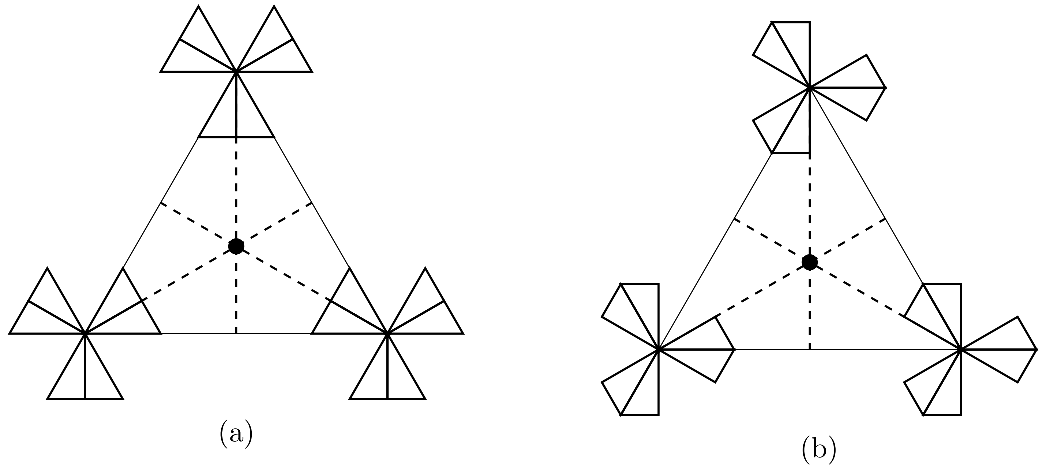


Figure 37: In both (a) and (b), the black dots in the middle are the rotation centers of the patterns. In (a), the 3 dashed lines are reflection axes; but in (b), the 3 dashed lines are not reflection axes. Therefore, the pattern (a) does not have off-axis rotations, and the pattern (b) has off-axis rotations.

7.1.1 Wallpaper Patterns With Parallelogram Unit Cells

Now we study the 17 types of wallpaper patterns classified by their unit cells. Given a wallpaper pattern with a generic parallelogram unit cell, the only symmetries of the point group that can map the lattice onto itself are the identity symmetry and rotation by π radians. Therefore, the point group of the symmetry group of a wallpaper pattern with generic parallelogram unit cell is either C_1 or C_2.

First, we start with the simplest point group: C_1, which is the trivial group containing the identity only. When a wallpaper pattern has a symmetry group whose point group $\overline{G} = C_1$, the motif of this pattern has no rotation or reflection symmetries. The whole pattern only contains translations and the identity symmetry, which is rotation by 0. The symmetry group of a such wallpaper pattern is called $p1$. An example of a wallpaper pattern with symmetry group $p1$ is shown in Figure 38 (a). Note that the motif of this pattern is at each angle of the parallelogram. It is denoted using one triangle. Recall that each triangle is a petal of the motif as defined in the previous chapter, and hence the motif of $p1$ only contains 1 petal.

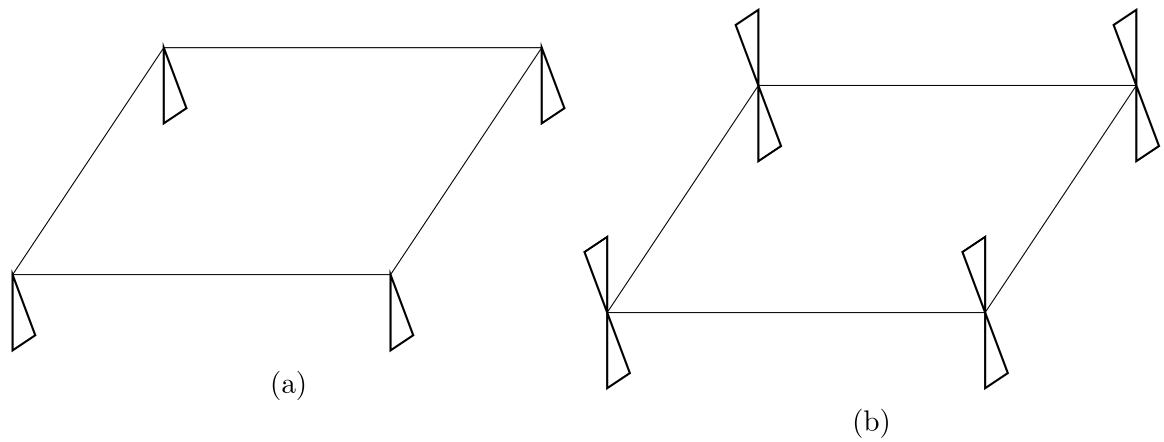


Figure 38: (a) A $p1$ wallpaper pattern; (b) A $p2$ wallpaper pattern.

When a wallpaper pattern has a symmetry group whose point group $\overline{G} = C_2$, a new wallpaper group $p2$ is generated. The point group of the wallpaper group contains identity and rotation by π. The symmetries of the whole wallpaper group $p2$ are translation, identity, and rotations by π. This forces the motif of a $p2$ wallpaper pattern to have rotation by π as a symmetry. Note that since there are no reflection axes in this pattern, all rotations in the pattern are o-rotations. An example of a wallpaper pattern with symmetry group $p2$ is shown in Figure 38 (b). The motif of this pattern is formed with 2 triangles, which indicates that the motif of $p2$ contains 2 petals.

7.1.2 Wallpaper Patterns With Rectangular Unit Cells

When the angle between the two translation axes is $\pi/2$, the unit cells of wallpaper patterns become rectangles. There are 5 types of rectangular wallpaper patterns and 5 wallpaper groups corresponding to them. When the unit cells are rectangular, there are reflection symmetries in the point groups of these wallpaper groups. Also, the point groups do not contain any rotation symmetries of order greater or equal to 3.

Therefore, the possible point groups are D_2 and D_4.

The point group of the first two rectangular wallpaper patterns is $\overline{G} = D_2 = \{E, H\}$, where H is single-axis reflection. Recall that the crystallographic group G has some isometry A such that $\phi(A) = H$ and we can have two cases depending on whether A is a reflection or a glide reflection. When A is a single-axis reflection, this wallpaper group G is called $p1m$. The point group of this wallpaper group contains identity and single-axis reflection. The symmetries of the whole wallpaper group $p1m$ are translation, identity, and single-axis reflection. This forces the motif of a $p1m$ wallpaper pattern to have single-axis reflection as a symmetry. An example of a wallpaper pattern with symmetry group $p1m$ is shown in Figure 39 (a). The motif of this pattern is formed with 2 triangles, indicating that the motif of a $p1m$ pattern has 2 petals.

When the point group is $\overline{G} = D_2 = \{E, H\}$ and A is a glide reflection, then the wallpaper group G is called $p1g$. The point group of this wallpaper group contains identity and glide reflection. The symmetries of the whole wallpaper group $p1g$ are translation, identity, and glide reflection. This forces the motif of a $p1g$ wallpaper pattern to have glide reflection as a symmetry. An example of a wallpaper pattern with symmetry group $p1g$ is shown in Figure 39 (b). The part contained by the dashed line is the motif of this pattern. The motif is formed with 2 triangles, indicating that the motif of a $p1g$ pattern has 2 petals.

When the point group is $\overline{G} = D_4 = \{E, R, H, V\}$, where R is rotation by π, H is 1-axis reflection, and V is 2-axis reflection. Suppose the crystallographic group G has isometries A_1, A_2 such that $\phi(A_1) = H$ and $\phi(A_2) = V$. Then A_1 and A_2 can be both reflections, both glide reflections, or one reflection and the other glide reflection. First, we let both A_1 and A_2 be reflections, then the wallpaper group G is called $p2mm$. The point group of this wallpaper group contains identity, rotation

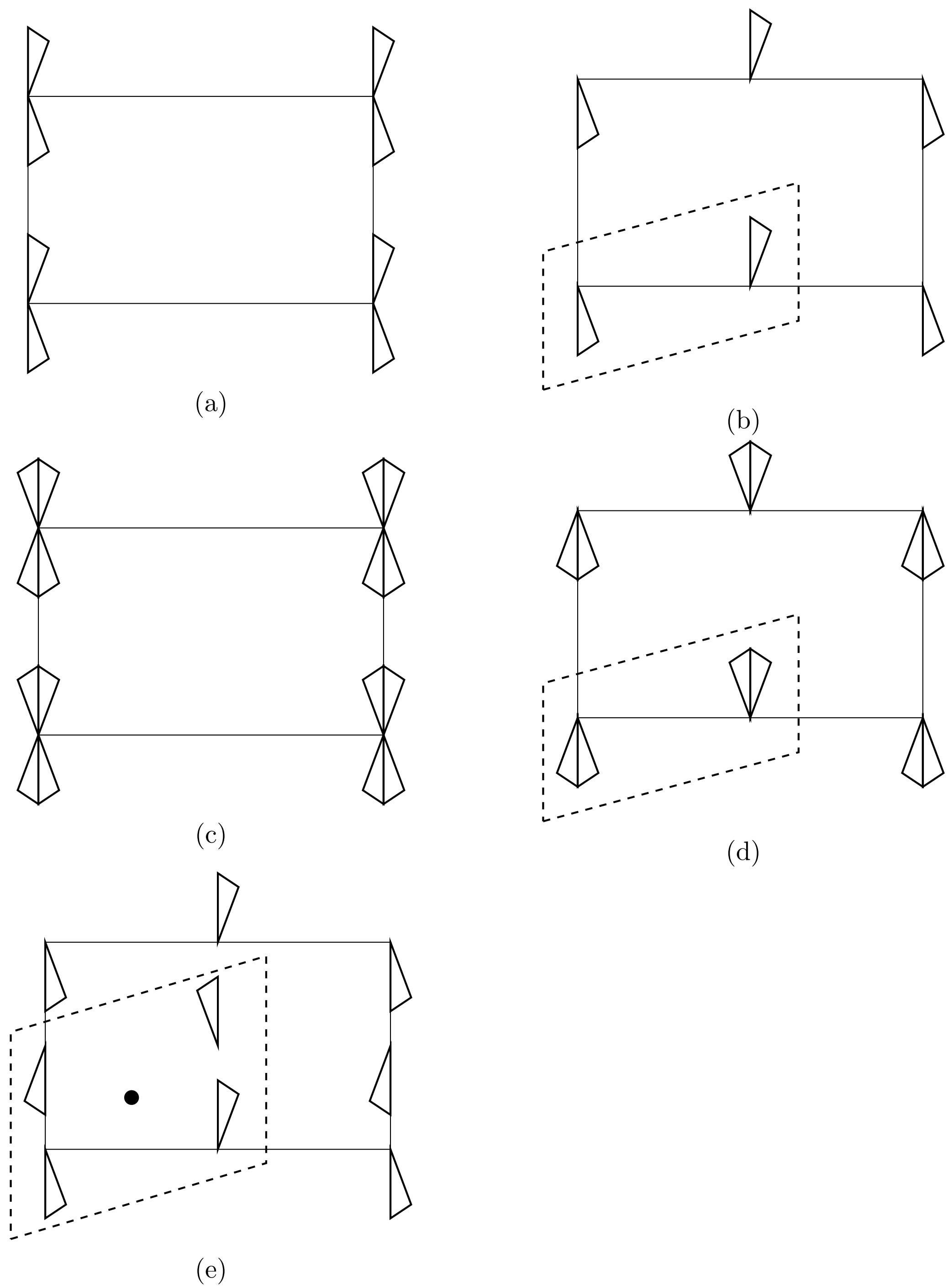


Figure 39: (a) A $p1m$ wallpaper pattern; (b) A $p1g$ wallpaper pattern; (c) A $p2mm$ wallpaper pattern; (d) A $p2gm$ wallpaper pattern; (e) A $p2gg$ wallpaper pattern.

by π, 1-axis reflection, and 2-axis reflection. The symmetries of the whole wallpaper group $p2mm$ are translation, identity, rotation by π, 1-axis reflection, and 2-axis reflection. This forces the motif of a $p2mm$ wallpaper pattern to have rotation by π, and reflection with two reflection axes as symmetries. An example of a wallpaper pattern with symmetry group $p2mm$ is shown in Figure 39 (c). The motif of this pattern is formed with 4 triangles, indicating that the motif of a $p2mm$ pattern has 4 petals.

Next, without loss of generality, if A_1 is a single-axis reflection and A_2 is a glide reflection, then the wallpaper group G is called $p2gm$. The point group of this wallpaper group contains identity, rotation by π, single-axis reflection, and glide reflection. The symmetries of the whole wallpaper group $p2gm$ are translation, identity, rotation by π, single-axis reflection, and glide reflection. This forces the motif of a $p2gm$ wallpaper pattern to have rotation by π, and single-axis reflection and glide reflection as symmetries. An example of a wallpaper pattern with symmetry group $p2gm$ is shown in Figure 39 (d). The part contained by the dashed line is the motif of this pattern. The motif is formed with 4 triangles, indicating that the motif of a $p2gm$ pattern has 4 petals.

If both A_1 and A_2 are glide reflections with different glide reflection axes, then the wallpaper group G is called $p2gg$. The point group of this wallpaper group contains identity, rotation by π, 1-axis glide reflection, and 2-axis glide reflection. The symmetries of the whole wallpaper group $p2gg$ are translation, identity, rotation by π, 1-axis glide reflection, and 2-axis glide reflection. This forces the motif of a $p2mm$ wallpaper pattern to have rotation by π, and glide reflection with two glide reflection axes as symmetries. An example of a wallpaper pattern with symmetry group $p2gg$ is shown in Figure 39 (e). The part contained by the dashed line is the motif of this pattern. The dot in the middle of the motif is the center of rotation of the whole pattern. The motif of this pattern is formed with 4 triangles, indicating that the motif of a $p2gg$ pattern also has 4 petals.

7.1.3 Wallpaper Patterns With Rhombic Unit Cells

When the length of the two translation vectors are equal, but the angle between the two translation vectors is not $\pi/2$. The unit cells of the wallpaper patterns are rhombic. There are 2 types of rhombic wallpaper patterns and 2 wallpaper groups corresponding to them. When the unit cells are rhombic, there are reflection symmetries in the point groups of these wallpaper groups. Also, the point groups do not contain any rotation symmetries of order greater or equal to 3. Therefore, the possible point groups are also D_2 and D_4. The rhombic wallpaper groups have the same point groups as rectangular wallpaper groups, but the rhombic wallpaper patterns are different from rectangular wallpaper patterns because their reflection axes are not parallel to any of the two translation vectors.

The point group of the first wallpaper group is $\overline{G} = D_2 = \{E, H\}$, and this wallpaper group is called $c1m$. The point group of this wallpaper group contains identity, single-axis reflection. The symmetries of the whole wallpaper group $c1m$ are translation, identity, and single-axis reflection. The reflection axes are not parallel to any of the two translation vecters. Instead, the reflection axes is in the middle of the two translation vecters. The motif of a $c1m$ wallpaper pattern has single-axis reflection as a symmetry. An example of a wallpaper pattern with symmetry group $c1m$ is shown in Figure 40 (a). The motif of this pattern is formed with 2 triangles, indicating that the motif of a $c1m$ pattern has 2 petals.

When the point group of a wallpaper group is $\overline{G} = D_4$, the wallpaper group G is called $c2mm$. The point group of this wallpaper group contains identity, rotations by π, 1-axis reflection, 2-axis reflection. The symmetries of the whole wallpaper group

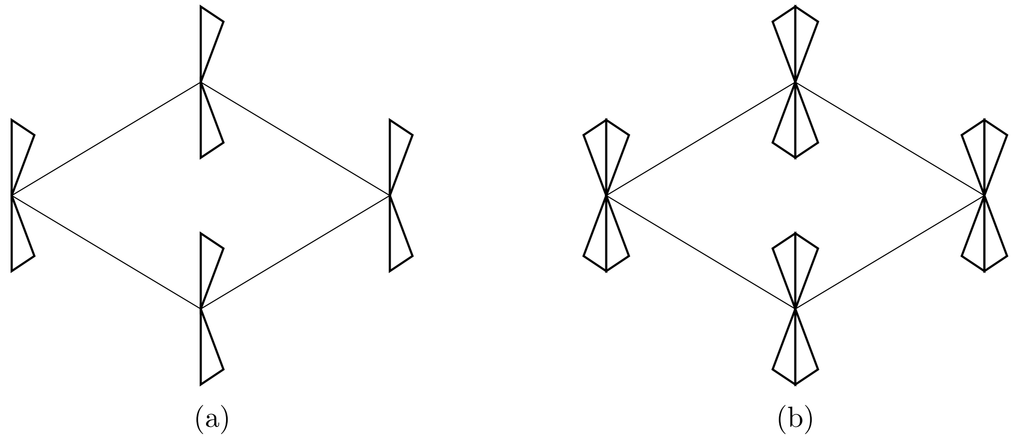


Figure 40: (a) A $c1m$ wallpaper pattern; (b) A $c2mm$ wallpaper pattern.

$c2mm$ are translation, identity, rotation by π, 2 reflections. The motif of a $c2mm$ wallpaper pattern has identity, rotation by π, and 2 reflections as symmetries. An example of a wallpaper pattern with symmetry group $c2mm$ is shown in Figure 40 (b). The motif of this pattern is formed with 4 triangles, indicating that the motif of a $c2mm$ pattern has 4 petals.

7.1.4 Wallpaper Patterns With Square Unit Cells

If the unit cells of wallpaper patterns are squares, there are 3 types of square wallpaper patterns and 3 wallpaper groups corresponding to them. When the unit cells are squares, C_4 rotation symmetries are contained in the point groups of the square wallpaper groups. Reflections may also be contained in the patterns. Therefore, the possible point groups for square wallpaper patterns are C_4 and D_8.

When the point group of a wallpaper group is $\overline{G} = C_4$, the wallpaper group G is called $p4$. The point group of this wallpaper group contains rotations by 0, $\pi/2$, π, and $3\pi/2$ radians. The symmetries of the whole wallpaper group $p4$ are translation and rotations by 0, $\pi/2$, π, and $3\pi/2$ radians. This forces the motif of a $p4$ wallpaper pattern to have all four rotation symmetries. Note that since there are no reflection axes in this pattern, all rotations in the pattern are o-rotations. An example of a wallpaper pattern with symmetry group $p4$ is shown in Figure 41 (a). The motif of this pattern is formed with 4 triangles, indicating that the motif of a $p4$ pattern has 4 petals.

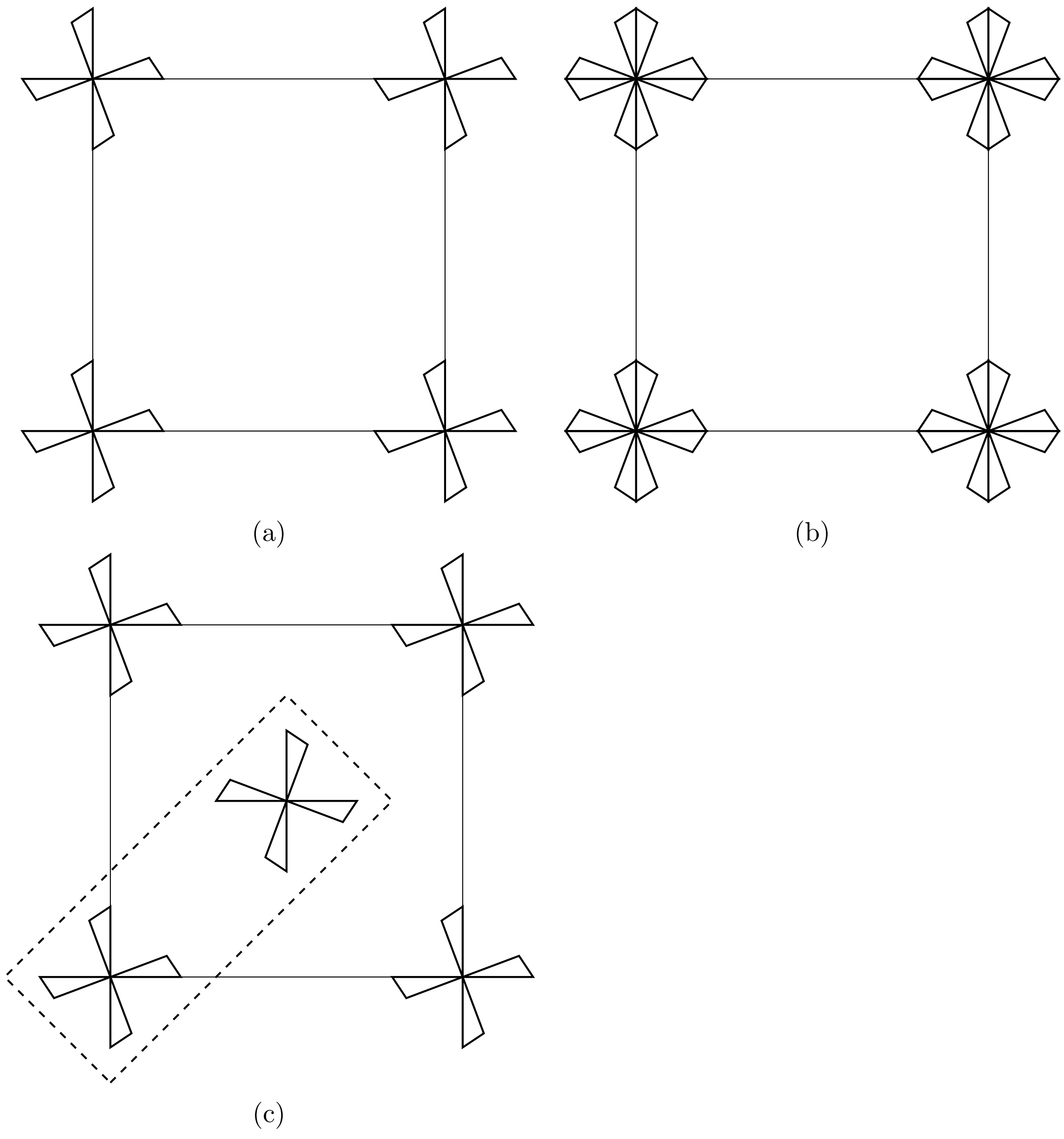


Figure 41: (a) A $p4$ wallpaper pattern; (b) A $p4mm$ wallpaper pattern; (c) A $p4gm$ wallpaper pattern.

When the point group of a wallpaper group is $\overline{G} = D_8$, which contains rotations by 0, $\pi/2$, π, and $3\pi/2$ radians, and 1-axis reflection H, 2-axis reflection V and two diagonal reflections. Again, suppose the crystallographic group G has isometries A_1, A_2 such that $\phi(A_1) = H$ and $\phi(A_2) = V$. If both A_1 and A_2 are reflections, then the wallpaper group G is called $p4mm$. The point group of this wallpaper group contains rotations by 0, $\pi/2$, π, and $3\pi/2$ radians, 1-axis reflection, 2-axis reflection, and two diagonal reflections. The symmetries of the whole wallpaper group $p4mm$ are translation, 4 rotation symmetries and 4 reflection symmetries. This forces the motif of a $p4mm$ wallpaper pattern to a D_8 symmetry group. An example of a wallpaper pattern with symmetry group $p4mm$ is shown in Figure 41 (b). The motif of this pattern is formed with 8 triangles, indicating that the motif of a $p4mm$ pattern has 8 petals.

If both A_1 and A_2 are glide reflections with different glide reflection axes, then the wallpaper group G is called $p4gm$. The point group of this wallpaper group contains rotations by 0, $\pi/2$, π, and $3\pi/2$, two diagonal reflections, 1-axis glide reflection, and 2-axis glide reflection. The symmetries of the whole wallpaper group $p4gm$ are translation, 4 rotation symmetries, 2 diagonal reflections, 1-axis glide reflection, and 2-axis glide reflection. This forces the motif of a $p4gm$ wallpaper pattern to have rotations by 0, $\pi/2$, π, and $3\pi/2$, two diagonal reflections, 1-axis glide reflection, and 2-axis glide reflection as symmetries. An example of a wallpaper pattern with symmetry group $p4gm$ is shown in Figure 41 (c). The part contained by the dashed line is the motif of this pattern. The motif is formed with 8 triangles, indicating that the motif of a $p4gm$ pattern also has 8 petals.

Note that since the rotation order of a wallpaper pattern with square unit cell is 4, we cannot have A_1 reflection and A_2 glide reflection.

7.1.5 Wallpaper Patterns With Hexagonal Unit Cells

If the unit cells of wallpaper patterns are hexagonal, there are 5 types of hexagonal wallpaper patterns and 5 wallpaper groups corresponding to them. When the unit cells are hexagonal, the point groups of these wallpaper patterns has rotation symmetries of order 3 or 6. Reflections may also be contained in the patterns. Therefore, the possible point groups for square wallpaper patterns are C_3, C_6, D_6 and D_{12}.

When the point group of a wallpaper group is $\overline{G} = C_3$, the wallpaper group G is called $p3$. The point group of this wallpaper group contains rotations by 0, $2\pi/3$, and $4\pi/3$ radians. The symmetries of the whole wallpaper group $p3$ are translation and rotations by 0, $2\pi/3$, and $4\pi/3$ radians. This forces the motif of a $p3$ wallpaper pattern to have all three rotation symmetries. Since there are no reflection axes in this pattern, all rotations in the pattern are o-rotations. An example of a wallpaper pattern with symmetry group $p3$ is shown in Figure 42 (a). The motif of this pattern is formed with 3 triangles, indicating that the motif of a $p3$ pattern has 3 petals.

When the point group of a wallpaper group is $\overline{G} = D_6$, which contains rotations by 0, $2\pi/3$, and $4\pi/3$ radians, and 3 reflections with 3 non-parallel reflection axes, we can have two wallpaper groups depending on whether o-rotation is present in the pattern. The wallpaper patterns that do not have o-rotations are called $p3m1$. Then all centers of rotation in the pattern are on reflection axes. The point group of this wallpaper group contains 3 rotations and 3 reflections. The symmetries of the whole wallpaper group $p3m1$ are translation, 3 rotations and 3 reflections. This forces the motif of a $p3m1$ wallpaper pattern to a D_6 symmetry group. An example of a wallpaper pattern with symmetry group $p3m1$ is shown in Figure 42 (b). The motif of this pattern is formed with 6 triangles, indicating that the motif of a $p3m1$ pattern has 6 petals.

When o-rotations are added in the wallpaper pattern with D_6 point group, the

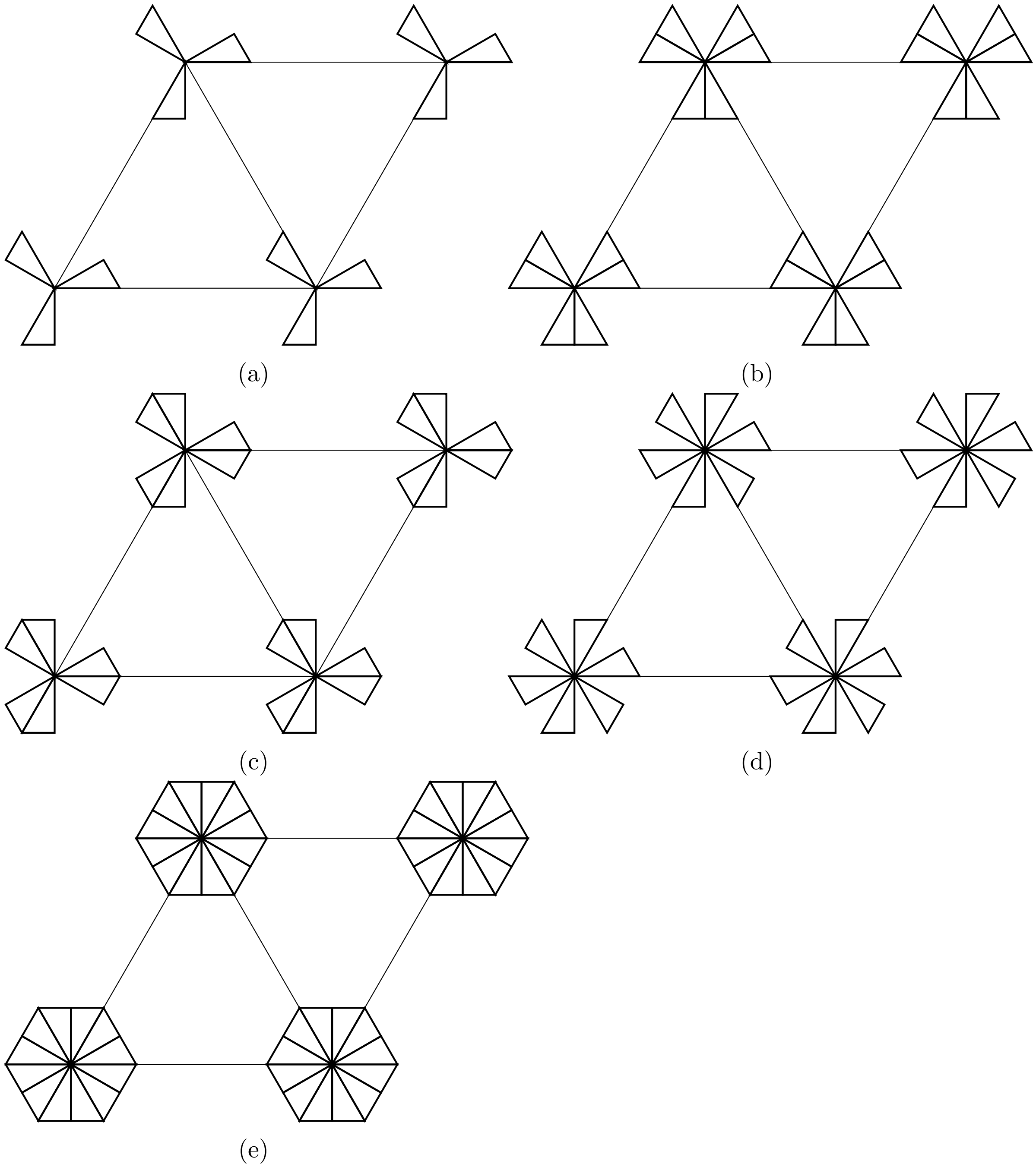


Figure 42: (a) A $p3$ wallpaper pattern; (b) A $p3m1$ wallpaper pattern; (c) A $p31m$ wallpaper pattern; (d) A $p6$ wallpaper pattern; (e) A $p6mm$ wallpaper pattern.

wallpaper groups are called $p31m$. Then not all centers of rotation in the pattern are on reflection axes. The point group of this wallpaper group also contains 3 rotations and 3 reflections. The symmetries of the whole wallpaper group $p31m$ are translation, 3 rotations and 3 reflections. This forces the motif of a $p31m$ wallpaper pattern to a D_6 symmetry group. An example of a wallpaper pattern with symmetry group $p31m$ is shown in Figure 42 (c). The motif of this pattern is formed with 6 triangles, indicating that the motif of a $p31m$ pattern has 6 petals.

If the point group of a wallpaper group is $\overline{G} = C_6$, then the wallpaper group G is called $p6$. The point group of this wallpaper group contains rotations by 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$ radians. The symmetries of the whole wallpaper group $p6$ are translation and rotations by 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$ radians. This forces the motif of a $p6$ wallpaper pattern to have all six rotation symmetries. Since there are no reflection axes in this pattern, all rotations in the pattern are o-rotations. An example of a wallpaper pattern with symmetry group $p6$ is shown in Figure 42 (d). The motif of this pattern is formed with 6 triangles, indicating that the motif of a $p6$ pattern has 6 petals.

Finally, the point group of the last wallpaper group is $\overline{G} = D_{12}$. This wallpaper group is called $p6mm$. The point group of this wallpaper group contains rotations by 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$ radians, and 6 reflections. No rotations in this wallpaper group are o-rotations. The symmetries of the whole wallpaper group $p6mm$ are translation, 6 rotations, and 6 reflections. This forces the motif of a $p6mm$ wallpaper pattern to have a D_{12} symmetry. An example of a wallpaper pattern with symmetry group $p6mm$ is shown in Figure 42 (e). The motif of this pattern is formed with 12 triangles, which indicates that the motif of a $p6mm$ pattern has 12 petals.

The following table is a summary of the point groups and unit cell types of the 17 wallpaper groups including the number of petals of a motif of each wallpaper pattern.

Point Group	Wallpaper Group	Unit Cell	# of Petals
C_1	$p1$	Parallelogram	1
C_2	$p2$	Parallelogram	2
C_3	$p3$	Hexagonal	3
C_4	$p4$	Square	4
C_6	$p6$	Hexagonal	6
D_2	$p1m$	Rectangular	2
D_2	$p1g$	Rectangular	2
D_2	$c1m$	Rhombic	2
D_4	$p2mm$	Rectangular	4
D_4	$p2gm$	Rectangular	4
D_4	$p2gg$	Rectangular	4
D_4	$c2mm$	Rhombic	4
D_6	$p3m1$	Hexagonal	6
D_6	$p31m$	Hexagonal	6
D_8	$p4mm$	Square	8
D_8	$p4gm$	Square	8
D_{12}	$p6mm$	Hexagonal	12

Table 5: Summary of the point group and unit cell type of each wallpaper group, and number of petals in a motif of each wallpaper pattern.

7.2 Future Work: Colorings of Wallpaper Patterns

As discussed in Section 6.2, each petal of a motif of a wallpaper pattern represents a fundamental region of a design whose symmetry group is a wallpaper group. Each petal cannot be subdivided. When we consider the perfect colorings of a motif of a wallpaper pattern, we use the subgroups of the point group of the corresponding wallpaper group and apply the scheme we introduced in Chapter 3. We can proceed as with frieze patterns and consider regular n-gons as motifs.

For example, the motif of a $p6mm$ wallpaper pattern has D_{12} symmetry, then we can use all perfect colorings of regular hexagons from Section 4.2 to perfect color a $p6mm$ wallpaper pattern. Some examples of perfect colorings of the $p6mm$ wallpaper pattern are shown in figure 43. We can also use perfect colorings of regular n-gons, where 6 is a divisor of n, to perfect coloring a $p6mm$ wallpaper pattern.

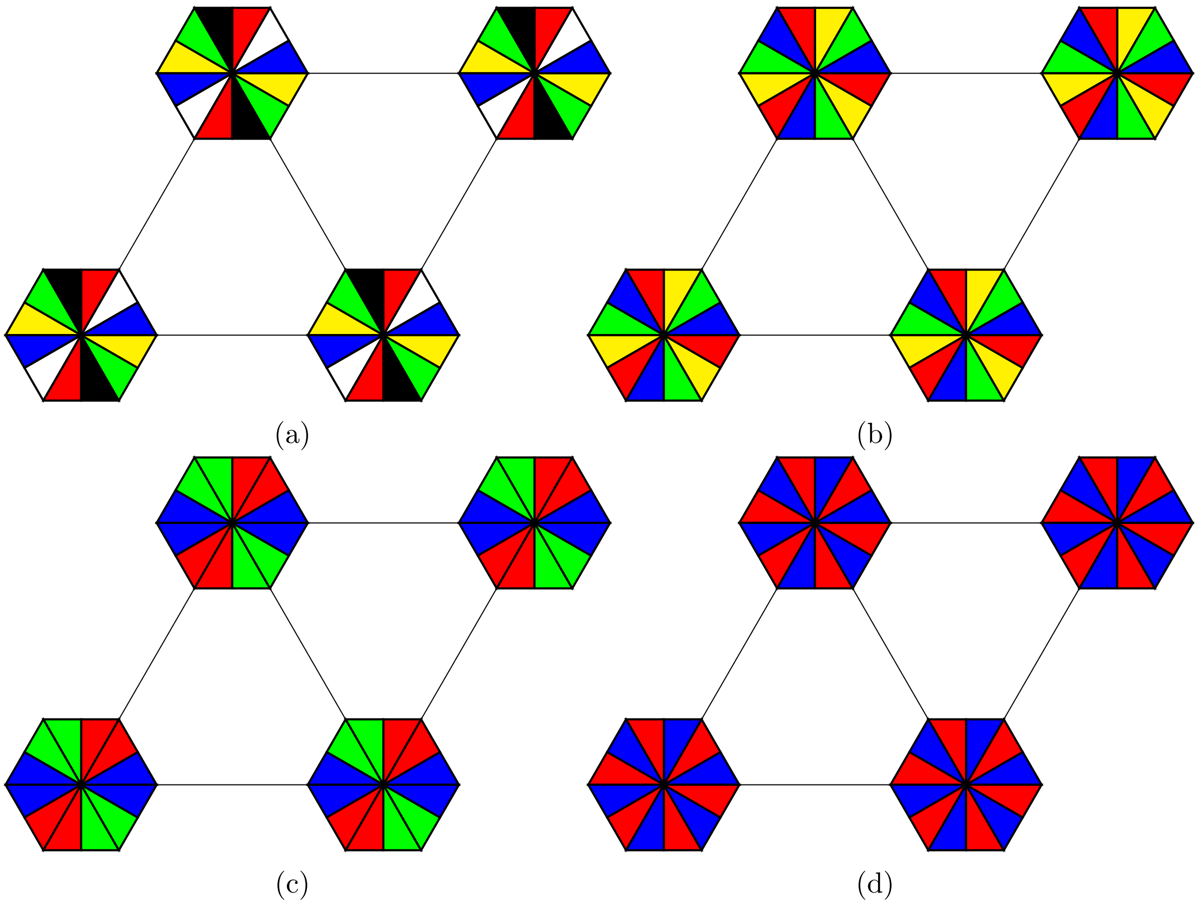


Figure 43: (a) A perfect 6-coloring of the $p6mm$ wallpaper pattern; (b) A perfect 4-coloring of the $p6mm$ wallpaper pattern; (c) A perfect 3-coloring of the $p6mm$ wallpaper pattern; (d) A perfect 2-coloring of the $p6mm$ wallpaper pattern.

More investigations are needed to construct theorems analogous to Theorem 6.2, Theorem 6.4, and Theorem 6.6, and to conclude the number of perfect colorings which can be applied to the motif for each wallpaper pattern. The approaches are similar to the study of the frieze patterns. We identify fundamental regions with petals and then consider the number of subgroups of D_{2n} containing certain elements.

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